Blow-up of Solutions to the Generalized Inviscid Proudman-Johnson Equation

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Abstract. For arbitrary values of a parameter $\lambda \in \mathbb{R}$, finite-time blow-up of solutions to the generalized, inviscid Proudman-Johnson equation is studied via a direct approach which involves the derivation of representation formulae for solutions to the problem.

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1. Introduction

In this article, we examine blow-up, and blow-up properties, in solutions to the initial boundary value problem

$$\begin{cases} u_{xt} + uu_{xx} - \lambda u_x^2 = I(t), & t > 0, \\ u(x,0) = u_0(x), & x \in [0,1], \\ I(t) = -(\lambda+1)\int_0^1 u_x^2\, dx, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, and solutions are subject to periodic boundary conditions

$$u(0,t) = u(1,t), \quad u_x(0,t) = u_x(1,t). \tag{1.2}$$

Equations (1.1)i), iii) may be obtained by integrating the partial differential equation

$$u_{xxt} + uu_{xxx} + (1-2\lambda)u_x u_{xx} = 0 \tag{1.3}$$

and using (1.2) ([19], [4], [17])[1]. We refer to (1.1) as the generalized, inviscid, Proudman-Johnson equation and note that the equation occurs in several different contexts, either with or without the nonlocal term $I(t)$. For $\lambda = -1$, it reduces to Burgers' equation. If $\lambda = -1/2$, the Hunter Saxton (HS) equation describes the orientation of waves in massive director nematic liquid crystals ([13], [2], [8], [24]). For periodic functions, the HS-equation also describes geodesics on the group $\mathcal{D}(\mathbb{S})\backslash Rot(\mathbb{S})$ of orientation preserving diffeomorphisms on the unit circle $\mathbb{S} = \mathbb{R}\backslash\mathbb{Z}$,

[1]Equation (1.3) was introduced in [17] with a parameter $a \in \mathbb{R}$ instead of the term $2\lambda - 1$. Since in this article we will be concerned with equation (1.1)i), iii), our choice of the parameter λ over a is due, mostly, to notational convenience.

modulo the subgroup of rigid rotations with respect to the right-invariant metric $\langle f, g\rangle = \int_{\mathbb{S}} f_x g_x dx$ ([15], [2], [21], [16]). If $\lambda = \frac{1}{n-1}$, $n \geq 2$, (1.1) i), iii) can be obtained directly from the $n-$dimensional incompressible Euler equations

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p, \qquad \nabla\cdot\boldsymbol{u} = 0$$

using stagnation point form velocities $\boldsymbol{u}(x, \boldsymbol{x}', t) = (u(x,t), -\lambda \boldsymbol{x}' u_x(x,t))$ for $\boldsymbol{x}' = \{x_2, ..., x_n\}$, or through the cylindrical coordinate representation $u^r = -\lambda r u_x(x,t)$, $u^\theta = 0$ and $u^x = u(x,t)$, where $r = |\boldsymbol{x}'|$, ([4], [22], [20], [17], [10]). Finally, in the local case $I(t) = 0$, the equation appears as a special case of Calogero's equation

$$u_{xt} + uu_{xx} - \Phi(u_x) = 0$$

for arbitrary functions $\Phi(\cdot)$ ([3]). The earliest results on blow-up in the nonlocal case $I(t) = -2\int_0^1 u_x^2 dx$ for $\lambda = 1$ are due to Childress et al. ([4]), where the authors show that there are blow-up solutions under Dirichlet boundary conditions. For spatially periodic solutions, the following is known:

- If $\lambda \in [-1/2, 0)$ and $u_0(x) \in W^{1,2}_{\mathbb{R}}(0,1)$, u_x remains bounded in the L^2 norm but blows up in the L^∞ norm ([18]). For $\lambda \in [-1, 0)$, if $u_0(x) \in H^s_{\mathbb{R}}(0,1)$, $s \geq 3$ and u_0'' is not constant, $\|u_x\|_\infty$ blows up ([23]), similarly if $\lambda \in (-2, -1)$, as long as
$$\inf_{x\in[0,1]}\{u_0'(x)\} + \sup_{x\in[0,1]}\{u_0'(x)\} < 0. \tag{1.4}$$
- For $\lambda \in (-\infty, -1/2)$, $\|u_x\|_2$ blows up in finite-time as long as ([17])
$$\int_0^1 u_0'(x)^3 dx < 0. \tag{1.5}$$
- If $\lambda \in [0, 1/2)$ and $u_0''(x) \in L^{\frac{1}{1-2\lambda}}_{\mathbb{R}}(0,1)$, u exists globally in time. Similarly, for $\lambda = 1/2$ as long as $u_0(x) \in W^{2,\infty}_{\mathbb{R}}(0,1)$ ([18], [20]).
- If $\lambda \in [1/2, 1)$ and $u_0'''(x) \in L^{\frac{1}{2(1-\lambda)}}_{\mathbb{R}}(0,1), u$ exists globally in time ([18]).

The purpose of this paper is to provide further insight on how periodic solutions to (1.1) blow up for parameters $\lambda \in (-\infty, 0)$ as well as to study regularity under differing assumptions on initial data when $\lambda \in [0, +\infty)$. To do this, we will examine solutions arising out of several classes of periodic, mean zero, initial data: the first, a class of smooth functions $u_0(x) \in C^\infty_{\mathbb{R}}(0,1)$, and then two classes of data for which either $u_0'(x)$ or $u_0''(x) \in PC_{\mathbb{R}}(0,1)$, the family of piecewise constant functions. The results are obtained via a direct approach which will involve the derivation of representation formulae for u_x along characteristics. The rest of the paper is organized as follows. A brief summary of new blow-up results is given in §2. The derivation of the solution representation formulae and proofs of the results are given in §3 and §4, respectively, as well as in appendix A. Finally, some illustrative examples are to be found in §5.

2. Summary of Results

Our first aim will be to obtain the representation formula, (3.19), for solutions to (1.1)-(1.2), which will permit us to estimate their lifetime for arbitrary $\lambda \in \mathbb{R}$. Given

$\eta_* \in \mathbb{R}^+$, to be defined, blow-up of solutions will depend upon the existence of a finite, positive, limit t_* defined by

$$t_* \equiv \lim_{\eta \uparrow \eta_*} \int_0^{\eta} \left(\int_0^1 \frac{d\alpha}{(1 - \lambda \mu u_0'(\alpha))^{\frac{1}{\lambda}}} \right)^{2\lambda} d\mu. \tag{2.1}$$

Let us suppose a solution $u(x,t)$ exists on an interval $t \in [0,T]$, $T < t_*$. Denote by $\gamma(\alpha,t)$ the solution to the initial value problem

$$\dot{\gamma}(\alpha,t) = u(\gamma(\alpha,t),t), \qquad \gamma(\alpha,0) = \alpha \in [0,1], \tag{2.2}$$

and define

$$M(t) \equiv \sup_{\alpha \in [0,1]} \{u_x(\gamma(\alpha,t),t)\}, \qquad M(0) = M_0, \tag{2.3}$$

and

$$m(t) \equiv \inf_{\alpha \in [0,1]} \{u_x(\gamma(\alpha,t),t)\}, \qquad m(0) = m_0, \tag{2.4}$$

where, for $u_0(\alpha) \in C^{\infty}_{\mathbb{R}}(0,1)$ and $\lambda > 0$, we will assume that the mean-zero function u_0' attains its greatest value $M_0 > 0$ at, at most, finitely many locations $\overline{\alpha}_i \in [0,1]$, $1 \le i \le m$. Similarly, for $\lambda < 0$, we suppose that the least value, $m_0 < 0$, occurs at a discrete set of points[2] $\underline{\alpha}_j \in [0,1]$, $1 \le j \le n$. From the above definitions and the solution formula, it can easily be shown that (see appendix C)

$$M(t) = u_x(\gamma(\overline{\alpha}_i,t),t), \qquad m(t) = u_x(\gamma(\underline{\alpha}_j,t),t). \tag{2.5}$$

The main results of this paper are summarized in the following theorems and in Corollary 2.9 below.

Theorem 2.6. *Consider the initial boundary value problem (1.1)-(1.2) for the generalized, inviscid, Proudman-Johnson equation. There exist smooth, mean-zero initial data such that:*

1. *For $\lambda \in [0,1]$, solutions exist globally in time. Particularly, these vanish as $t \uparrow t_* = +\infty$ for $\lambda \in (0,1)$ but converge to a non-trivial steady-state if $\lambda = 1$.*
2. *For $\lambda \in \mathbb{R}\backslash(-2,1]$, there exists a finite $t_* > 0$ such that both the maximum $M(t)$ and the minimum $m(t)$ diverge to $+\infty$ and to $-\infty$, respectively, as $t \uparrow t_*$. In addition, for every $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$, $\lim_{t \uparrow t_*} |u_x(\gamma(\alpha,t),t)| = +\infty$ (two-sided, everywhere blow-up).*
3. *For $\lambda \in (-2,0)$, there is a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up).*

Subsequent results examine the behaviour, as $t \uparrow t_*$, of two quantities, the jacobian $\gamma_\alpha(\alpha,t)$ (see (2.2)), and the L^p norm

$$\|u_x(x,t)\|_p = \left(\int_0^1 (u_x(\gamma(\alpha,t),t))^p \gamma_\alpha(\alpha,t)\, d\alpha \right)^{1/p}, \quad p \in [1,+\infty), \tag{2.7}$$

with particular emphasis given to the energy function $E(t) = \|u_x\|_2^2$.

Remark 2.8. Corollary 2.9 and theorem 2.11 below describe pointwise behaviour and L^p-regularity of solutions as $t \uparrow t_*$ where, for $\lambda \in \mathbb{R}\backslash[0,1]$, $t_* > 0$ refers to the *finite* L^∞ blow-up time of theorem 2.6; otherwise the description is asymptotic, for $t \uparrow t_* = +\infty$.

[2] One possibility for admitting infinitely many $\overline{\alpha}_i$ and/or $\underline{\alpha}_j$ will be considered below for the class $PC_{\mathbb{R}}(0,1)$.

TABLE 1. Energy Estimates and L^p Regularity as $t \uparrow t_*$

λ	$E(t)$	$\dot{E}(t)$	u_x
$(-\infty, -2]$	$+\infty$	$+\infty$	$\notin L^p$, $p > 1$
$(-2, -2/3]$	$+\infty$	$+\infty$	$\in L^1$, $\notin L^2$
$(-2/3, -1/2)$	Bounded	$+\infty$	$\in L^2$, $\notin L^3$
$-1/2$	Constant	0	$\in L^2$, $\notin L^3$
$(-1/2, -2/5]$	Bounded	$-\infty$	$\in L^2$, $\notin L^3$
$\left(-\frac{2}{2p-1}, 0\right)$, $p \geq 3$	Bounded	Bounded	$\in L^p$
$\left[-\frac{2}{p-1}, -\frac{2}{p}\right]$, $p \geq 6$	Bounded	Bounded	$\notin L^p$
$[0, 1]$	Bounded	Bounded	$\in L^\infty$
$(1, +\infty)$	$+\infty$	$+\infty$	$\notin L^p$, $p > 1$

Corollary 2.9. *Let $u(x,t)$ in Theorem 2.6 be a solution to the initial boundary value problem (1.1)-(1.2) defined for $t \in [0, t_*)$. Then*

$$\lim_{t \uparrow t_*} \gamma_\alpha(\alpha, t) = \begin{cases} +\infty, & \alpha = \overline{\alpha}_i, \quad \lambda \in (0, +\infty), \\ 0, & \alpha \neq \overline{\alpha}_i, \quad \lambda \in (0, 2], \\ C, & \alpha \neq \overline{\alpha}_i, \quad \lambda \in (2, +\infty), \\ 0, & \alpha = \underline{\alpha}_j, \quad \lambda \in (-\infty, 0), \\ C, & \alpha \neq \underline{\alpha}_j, \quad \lambda \in (-\infty, 0) \end{cases} \tag{2.10}$$

for positive constants C which depend on the choice of λ and α.

Theorem 2.11. *Let $u(x,t)$ in Theorem 2.6 be a solution to the initial boundary value problem (1.1)-(1.2) defined for $t \in [0, t_*)$. It holds,*

1. *For $p \geq 1$ and $\frac{2}{1-2p} < \lambda \leq 1$, $\lim_{t \uparrow t_*} \|u_x\|_p < +\infty$.*
2. *For $p > 1$ and $\lambda \in \mathbb{R} \backslash (-2, 1]$, $\lim_{t \uparrow t_*} \|u_x\|_p = +\infty$. Similarly, for $p \in (1, +\infty)$ and $\lambda \in (-2, -2/p]$.*
3. *The energy $E(t) = \|u_x\|_2^2$ diverges if $\lambda \in \mathbb{R} \backslash (-2/3, 1]$ as $t \uparrow t_*$ but remains finite for $t \in [0, t_*]$ otherwise. Moreover, $\dot{E}(t)$ blows up to $+\infty$ as $t \uparrow t_*$ when $\lambda \in \mathbb{R} \backslash [-1/2, 1]$ and $\dot{E}(t) \equiv 0$ for $\lambda = -1/2$; whereas, $\lim_{t \uparrow t_*} \dot{E}(t) = -\infty$ if $\lambda \in (-1/2, -2/5]$ but remains bounded when $\lambda \in (-2/5, 1]$ for all $t \in [0, t_*]$.*

See Table 1 for a summary of the results mentioned in theorem 2.11.

Remark 2.12. Global weak solutions to (1.1)i) having $I(t) = 0$ and $\lambda = -1/2$ have been studied by several authors, ([14], [2], [16]). Such solutions have also been constructed for $\lambda \in [-1/2, 0)$ in [6] (c.f. also [5]) by extending an argument used in [2]. Notice that theorems 2.6 and 2.11 above imply the existence of smooth data and a finite $t_* > 0$ such that strong solutions to (1.1)-(1.2) with $\lambda \in (-2/3, 0)$ satisfy $\lim_{t \uparrow t_*} \|u_x\|_\infty = +\infty$ but $\lim_{t \uparrow t_*} E(t) < +\infty$. As a result, it is possible that the representation formulae derived in §3 can lead to similar construction of global, weak solutions for $\lambda \in (-2/3, 0)$.

The results stated thus far will be established for a family of smooth functions $u_0(x) \in C^\infty_{\mathbb{R}}(0,1)$ having, relative to the sign of λ, global extrema attained at finitely many points. If we next consider periodic $u_0'(x) \in PC_{\mathbb{R}}(0,1)$, the class of mean-zero, piecewise constant functions, the following holds instead:

Theorem 2.13. *For the initial boundary value problem (1.1)-(1.2) with $u_0'(\alpha) \in PC_{\mathbb{R}}(0,1)$ assume solutions are defined for all $t \in [0,T]$, $T > 0$. Then no $W^{1,\infty}(0,1)$ solution may exist for $T \geq t_*$, where $0 < t_* < +\infty$ if $\lambda \in (-\infty, 0)$, and $t_* = +\infty$ for $\lambda \in [0, +\infty)$. Further, $\lim_{t \uparrow t_*} \|u_x\|_1 = +\infty$ when $\lambda \in (-\infty, -1)$ while*

$$\lim_{t \uparrow t_*} \|u_x\|_p = \begin{cases} C, & -\frac{1}{p} \leq \lambda < 0, \\ +\infty, & -1 \leq \lambda < -\frac{1}{p}, \end{cases}$$

for $p \geq 1$ and where the positive constants C depend on the choice of λ and p.

Finally, the case of periodic $u_0'' \in PC_{\mathbb{R}}$ is briefly examined in §4.2.2 via a simple example. Our findings are summarized in theorem 2.14 below.

Theorem 2.14. *For the initial boundary value problem (1.1)-(1.2) with $u_0''(\alpha) \in PC_{\mathbb{R}}(0,1)$ and $\lambda \in \mathbb{R} \backslash [0, 1/2]$, there are blow-up solutions. Specifically, when $\lambda \in (1/2, +\infty)$, solutions can undergo a two-sided, everywhere blow-up in finite-time, whereas for $\lambda \in (-\infty, 0)$, divergence of the minimum to negative infinity can occur at a finite number of locations.*

Remark 2.15. In addition to providing an approach for the case $\lambda \in (1, +\infty)$ and giving a more detailed description of the L^p regularity of solutions, the advantage of having the solution formula (3.19) available is that conditions such as (1.4) and (1.5), though sufficient for blow-up, will not be necessary in our future arguments.

3. The General Solution

We now establish our solution formulae for (1.1)-(1.2). Given $\lambda \in \mathbb{R} \backslash \{0\}$, equations (1.1)i), iii) admit a second-order, linear, ordinary differential equation for the jacobian $\gamma_\alpha(\alpha, t)$. The case $\lambda = 0$ will be considered separately in appendix A. In the reformulated problem, a general solution is constructed which shows $u_x(\gamma(\alpha,t),t)$ to satisfy (1.1)i) along characteristics, namely

$$\frac{d}{dt}(u_x(\gamma(\alpha,t),t)) - \lambda u_x(\gamma(\alpha,t),t)^2 = I(t). \tag{3.1}$$

Since $\dot\gamma(\alpha,t) = u(\gamma(\alpha,t),t)$,

$$\dot\gamma_\alpha = (u_x \circ \gamma) \cdot \gamma_\alpha \tag{3.2}$$

therefore, using (1.1) and (3.2),

$$\begin{aligned} \ddot\gamma_\alpha &= (u_{xt} + u u_{xx}) \circ \gamma \cdot \gamma_\alpha + (u_x \circ \gamma) \cdot \dot\gamma_\alpha \\ &= (u_{xt} + u u_{xx}) \circ \gamma \cdot \gamma_\alpha + u_x^2 \circ \gamma \cdot \gamma_\alpha \\ &= (\lambda + 1) \left(u_x^2 \circ \gamma - \int_0^1 u_x^2 dx \right) \cdot \gamma_\alpha \\ &= (\lambda + 1) \left((\gamma_\alpha^{-1} \cdot \dot\gamma_\alpha)^2 - \int_0^1 u_x^2 dx \right) \cdot \gamma_\alpha \,. \end{aligned} \tag{3.3}$$

For $I(t) = -(\lambda+1)\int_0^1 u_x^2 dx$ and $\lambda \in \mathbb{R}\backslash\{0\}$, then

$$I(t) = \frac{\ddot{\gamma}_\alpha \cdot \gamma_\alpha - (\lambda+1)\cdot \dot{\gamma}_\alpha^2}{\gamma_\alpha^2} = -\frac{\gamma_\alpha^\lambda \cdot (\gamma_\alpha^{-\lambda})^{\cdot\cdot}}{\lambda} \tag{3.4}$$

and so

$$(\gamma_\alpha^{-\lambda})^{\cdot\cdot} + \lambda \gamma_\alpha^{-\lambda} I(t) = 0. \tag{3.5}$$

Setting

$$\omega(\alpha, t) = \gamma_\alpha(\alpha, t)^{-\lambda} \tag{3.6}$$

yields

$$\ddot{\omega}(\alpha, t) + \lambda I(t)\omega(\alpha, t) = 0, \tag{3.7}$$

an ordinary differential equation parametrized by α. Suppose we have two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$ to (3.7), satisfying $\phi_1(0) = \dot{\phi}_2(0) = 1$, $\dot{\phi}_1(0) = \phi_2(0) = 0$. Then by Abel's formula, $\mathrm{W}(\phi_1(t), \phi_2(t)) = 1$, $t \geq 0$, where $\mathrm{W}(g, h)$ denotes the wronskian of g and h. We look for solutions of (3.7), satisfying appropriate initial data, of the form

$$\omega(\alpha, t) = c_1(\alpha)\phi_1(t) + c_2(\alpha)\phi_2(t), \tag{3.8}$$

where reduction of order allows us to write $\phi_2(t)$ in terms of $\phi_1(t)$ as

$$\phi_2(t) = \phi_1(t)\int_0^t \frac{ds}{\phi_1^2(s)}.$$

Since $\dot{\omega} = -\lambda\gamma_\alpha^{-(\lambda+1)}\dot{\gamma_\alpha}$ by (3.6) and $\gamma_\alpha(\alpha, 0) = 1$, then $\omega(\alpha, 0) = 1$ and $\dot{\omega}(\alpha, 0) = -\lambda u_0'(\alpha)$, from which $c_1(\alpha)$ and $c_2(\alpha)$ are obtained. Combining these results reduces (3.8) to

$$\omega(\alpha, t) = \phi_1(t)\left(1 - \lambda\eta(t)u_0'(\alpha)\right), \qquad \eta(t) = \int_0^t \frac{ds}{\phi_1^2(s)}. \tag{3.9}$$

Now, (3.6) and (3.9) imply

$$\gamma_\alpha(\alpha, t) = \left(\phi_1(t)\mathcal{J}(\alpha, t)\right)^{-\frac{1}{\lambda}}, \tag{3.10}$$

where

$$\mathcal{J}(\alpha, t) = 1 - \lambda\eta(t)u_0'(\alpha), \qquad \mathcal{J}(\alpha, 0) = 1, \tag{3.11}$$

however, uniqueness of solution to (2.2) and periodicity of u require

$$\gamma(\alpha + 1, t) - \gamma(\alpha, t) = 1 \tag{3.12}$$

for as long as u is defined. Spatially integrating (3.10) therefore yields

$$\phi_1(t) = \left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{\frac{1}{\lambda}}}\right)^\lambda, \tag{3.13}$$

and so, if we set

$$\mathcal{K}_i(\alpha, t) = \frac{1}{\mathcal{J}(\alpha, t)^{i+\frac{1}{\lambda}}}, \qquad \bar{\mathcal{K}}_i(t) = \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha, t)^{i+\frac{1}{\lambda}}} \tag{3.14}$$

for $i = 0, 1, 2, ...$, we can write γ_α in the form

$$\gamma_\alpha = \mathcal{K}_0/\bar{\mathcal{K}}_0. \tag{3.15}$$

As a result of using (3.2) and (3.15), we obtain

$$u_x(\gamma(\alpha,t),t) = \dot{\gamma}_\alpha(\alpha,t)/\gamma_\alpha(\alpha,t) = (\ln(\mathcal{K}_0/\bar{\mathcal{K}}_0))^{\cdot}\,. \tag{3.16}$$

In addition, differentiating (3.9)ii) gives

$$\dot{\eta}(t) = \left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}}\right)^{-2\lambda}, \qquad \eta(0) = 0, \tag{3.17}$$

from which it follows that the existence of an eventual finite blow-up time $t_* > 0$ will depend, in part, upon convergence of the integral

$$t(\eta) = \int_0^\eta \left(\int_0^1 \frac{d\alpha}{(1-\lambda\mu u_0'(\alpha))^{\frac{1}{\lambda}}}\right)^{2\lambda} d\mu \tag{3.18}$$

as $\eta \uparrow \eta_*$ for $\eta_* > 0$ to be defined. In an effort to simplify the following arguments, we point out that (3.16) can be rewritten in a slightly more useful form. The result is

$$u_x(\gamma(\alpha,t),t) = \frac{1}{\lambda\eta(t)\bar{\mathcal{K}}_0(t)^{2\lambda}}\left(\frac{1}{\mathcal{J}(\alpha,t)} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\right). \tag{3.19}$$

This is derived as follows. From (3.14) and (3.16),

$$u_x(\gamma(\alpha,t),t) = \frac{1}{\bar{\mathcal{K}}_0(t)^{2\lambda}}\left(\frac{u_0'(\alpha)}{\mathcal{J}(\alpha,t)} - \frac{1}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}}\right). \tag{3.20}$$

However

$$\frac{u_0'(\alpha)}{\mathcal{J}(\alpha,t)} = \frac{1}{\lambda\eta(t)}\left(\frac{1}{\mathcal{J}(\alpha,t)} - 1\right), \tag{3.21}$$

by (3.11), and so

$$\int_0^1 \frac{u_0'(\alpha)\,d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}} = \frac{\bar{\mathcal{K}}_1(t) - \bar{\mathcal{K}}_0(t)}{\lambda\eta(t)}. \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.20) yields (3.19). Finally, assuming sufficient smoothness, we may use (3.15) and (3.19) to obtain ([20], [23])

$$u_{xx}(\gamma(\alpha,t),t) = u_0''(\alpha)(\gamma_\alpha(\alpha,t))^{2\lambda-1}. \tag{3.23}$$

Equation (3.23) implies that as long as a solution exists it will maintain its initial concavity profile. Also, since the exponent above changes sign through $\lambda = 1/2$, blow-up implies, relative to the value of λ, either vanishing or divergence of the jacobian. More explicitly, (3.15) and (3.23) yield

$$u_{xx}(\gamma(\alpha,t),t) = \frac{u_0''(\alpha)}{\mathcal{J}(\alpha,t)^{2-\frac{1}{\lambda}}}\left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}}\right)^{1-2\lambda}. \tag{3.24}$$

4. Global Estimates and Blow-up

In §4.1.1-4.1.3 we establish Theorem 2.6 and Corollary 2.9, while Theorem 2.11 is proved in §4.1.4. Theorems 2.13 and 2.14 are proved in §4.2.1 and §4.2.2, respectively.

For $M_0 > 0 > m_0$ as in (2.3) and (2.4), set

$$\eta_* = \begin{cases} \frac{1}{\lambda M_0}, & \lambda > 0, \\ \frac{1}{\lambda m_0}, & \lambda < 0. \end{cases} \tag{4.1}$$

Then, as $\eta \uparrow \eta_*$, the space-dependent term in (3.19) will diverge for certain choices of α and not at all for others. Specifically, for $\lambda > 0$, $\mathcal{J}(\alpha,t)^{-1}$ blows up earliest as $\eta \uparrow \eta_*$ at $\alpha = \overline{\alpha}_i$, since

$$\mathcal{J}(\overline{\alpha}_i,t)^{-1} = (1-\lambda\eta(t)M_0)^{-1} \to +\infty \quad \text{as} \quad \eta \uparrow \eta_* = \frac{1}{\lambda M_0}.$$

Similarly for $\lambda < 0$, $\mathcal{J}(\alpha,t)^{-1}$ diverges first at $\alpha = \underline{\alpha}_j$ and

$$\mathcal{J}(\underline{\alpha}_j,t)^{-1} = (1-\lambda\eta(t)m_0)^{-1} \to +\infty \quad \text{as} \quad \eta \uparrow \eta_* = \frac{1}{\lambda m_0}.$$

However, blow-up of (3.19) does not necessarily follow from this; we will need to estimate the behaviour of the time-dependent integrals

$$\bar{\mathcal{K}}_0(t) = \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}}, \qquad \bar{\mathcal{K}}_1(t) = \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}}$$

as $\eta \uparrow \eta_*$. To this end, in some of the proofs we find convenient the use of the Gauss hypergeometric series ([1], [9], [12])

$$ {}_2F_1\left[a,b;c;z\right] \equiv \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\;k!} z^k, \qquad |z|<1 \tag{4.2}$$

for $c \notin \mathbb{Z}^- \cup \{0\}$ and $(x)_k$, $k \in \mathbb{N} \cup \{0\}$, the Pochhammer symbol $(x)_0 = 1$, $(x)_k = x(x+1)...(x+k-1)$. Also, we will make use of the following results ([9], [12]):

Proposition 4.3. *Suppose $|arg(-z)| < \pi$ and $a,b,c,a-b \notin \mathbb{Z}$. Then, the analytic continuation for $|z|>1$ of the series (4.2) is given by*

$$\begin{aligned} {}_2F_1[a,b;c;z] =& \frac{\Gamma(c)\Gamma(a-b)(-z)^{-b}\,{}_2F_1[b,1+b-c;1+b-a;z^{-1}]}{\Gamma(a)\Gamma(c-b)} \\ &+ \frac{\Gamma(c)\Gamma(b-a)(-z)^{-a}\,{}_2F_1[a,1+a-c;1+a-b;z^{-1}]}{\Gamma(b)\Gamma(c-a)} \end{aligned} \tag{4.4}$$

where $\Gamma(\cdot)$ denotes the standard gamma function.

Lemma 4.5. *Suppose $b \in (-\infty,2)\backslash\{1/2\}$, $0 \le |\beta-\beta_0| \le 1$ and $\epsilon \ge C_0$ for some $C_0 > 0$. Then*

$$\frac{1}{\epsilon^b}\frac{d}{d\beta}\left((\beta-\beta_0)\,{}_2F_1\left[\frac{1}{2},b;\frac{3}{2};-\frac{C_0(\beta-\beta_0)^2}{\epsilon}\right]\right) = (\epsilon + C_0(\beta-\beta_0)^2)^{-b}. \tag{4.6}$$

Proof. See appendix B. □

4.1. A Class of Smooth Initial Data

In this section, we study finite-time blow-up of solutions to (1.1)-(1.2) which arise from a class of mean-zero, smooth data. In §4.1.1, we consider parameter values $\lambda \in [0,+\infty)$ whereas the case $\lambda \in (-\infty,0)$ is studied in §4.1.2 and §4.1.3. Finally, L^p regularity of solutions is examined in §4.1.4 for $p \in [1,+\infty)$.

4.1.1. Global estimates for $\lambda \in [0,1]$ and blow-up for $\lambda \in (1,+\infty)$.
In Theorem 4.7 below, we prove finite-time blow-up of u_x in the L^∞ norm for $\lambda \in (1,+\infty)$. In fact, we will find that the blow-up is two-sided and occurs everywhere in the domain, an event we will refer to as "two-sided, everywhere blow-up." In contrast, for parameters $\lambda \in [0,1]$, we show that solutions persist globally in time. More particularly, these vanish as $t \to +\infty$ for $\lambda \in (0,1)$ but converge to a non-trivial steady-state if $\lambda = 1$. Finally, the behaviour of the jacobian (3.15) is also studied. We refer to appendix A for the case $\lambda = 0$.

Theorem 4.7. *Consider the initial boundary value problem (1.1)-(1.2). There exist smooth, mean-zero initial data such that:*

1. *For $\lambda \in (0,1]$, solutions persist globally in time. In particular, these vanish as $t \uparrow t_* = +\infty$ for $\lambda \in (0,1)$ but converge to a non-trivial steady-state if $\lambda = 1$.*
2. *For $\lambda \in (1,+\infty)$, there exists a finite $t_* > 0$ such that both the maximum $M(t)$ and the minimum $m(t)$ diverge to $+\infty$ and respectively to $-\infty$ as $t \uparrow t_*$. Moreover, $\lim_{t\uparrow t_*} u_x(\gamma(\alpha,t),t) = -\infty$ for $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$ (two-sided, everywhere blow-up).*

Finally, for t_* as above, the jacobian (3.15) satisfies

$$\lim_{t\uparrow t_*} \gamma_\alpha(\alpha,t) = \begin{cases} +\infty, & \alpha = \overline{\alpha}_i, \quad \lambda \in (0,+\infty), \\ 0, & \alpha \neq \overline{\alpha}_i, \quad \lambda \in (0,2], \\ C, & \alpha \neq \overline{\alpha}_i, \quad \lambda \in (2,+\infty) \end{cases} \tag{4.8}$$

where the positive constants C depend on the choice of λ and $\alpha \neq \overline{\alpha}_i$.

Proof. For simplicity, assume $M_0 > 0$ is attained at a single location[3] $\overline{\alpha} \in (0,1)$. We consider the case where, near $\overline{\alpha}$, $u_0'(\alpha)$ has non-vanishing second order derivative, so that, locally $u_0'(\alpha) \sim M_0 + C_1(\alpha - \overline{\alpha})^2$ for $0 \leq |\alpha - \overline{\alpha}| \leq s$, $0 < s \leq 1$ and $C_1 = u_0'''(\overline{\alpha})/2 < 0$. Then, for $\epsilon > 0$

$$\epsilon - u_0'(\alpha) + M_0 \sim \epsilon - C_1(\alpha - \overline{\alpha})^2. \tag{4.9}$$

Global existence for $\lambda \in (0,1]$.
By (4.9) above and the change of variables $\alpha = \sqrt{\frac{\epsilon}{|C_1|}} \tan\theta + \overline{\alpha}$, we have that

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha-\overline{\alpha})^2)^{\frac{1}{\lambda}}} \sim \frac{\epsilon^{\frac{1}{2}-\frac{1}{\lambda}}}{\sqrt{-C_1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{2\left(\frac{1}{\lambda}-1\right)} d\theta \tag{4.10}$$

for $\epsilon > 0$ small and $\lambda \in (0,1]$. But from properties of the Gamma function (see for instance [11]), the identity

$$\int_0^1 t^{p-1}(1-t)^{q-1} dt = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \tag{4.11}$$

holds for all $p,q > 0$. Therefore, setting $p = \frac{1}{2}$, $q = \frac{1}{\lambda} - \frac{1}{2}$ and $t = \sin^2\theta$ into (4.11) gives

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(\theta))^{2\left(\frac{1}{\lambda}-1\right)} d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{\lambda}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{\lambda}\right)},$$

which we use, along with (4.9) and (4.10), to obtain

$$\int_0^1 \frac{d\alpha}{(\epsilon - u_0'(\alpha) + M_0)^{\frac{1}{\lambda}}} \sim \frac{\Gamma\left(\frac{1}{\lambda}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{\lambda}\right)} \sqrt{-\frac{\pi}{C_1}}\, \epsilon^{\frac{1}{2}-\frac{1}{\lambda}}. \tag{4.12}$$

[3]The case of a finite number of $\overline{\alpha}_i \in [0,1]$ follow similarly.

Consequently, setting $\epsilon = \frac{1}{\lambda\eta} - M_0$ into (4.12) yields

$$\bar{\mathcal{K}}_0(t) \sim C_3 \mathcal{J}(\overline{\alpha}, t)^{\frac{1}{2}-\frac{1}{\lambda}} \tag{4.13}$$

for $\eta_* - \eta > 0$ small, $\mathcal{J}(\overline{\alpha}, t) = 1 - \lambda\eta(t)M_0$, $\eta_* = \frac{1}{\lambda M_0}$ and positive constants C_3 given by

$$C_3 = \frac{\Gamma\left(\frac{1}{\lambda} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{\lambda}\right)} \sqrt{-\frac{\pi M_0}{C_1}}. \tag{4.14}$$

Similarly,

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha - \overline{\alpha})^2)^{1+\frac{1}{\lambda}}} \sim \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\lambda}\right)}{\Gamma\left(1 + \frac{1}{\lambda}\right)} \sqrt{-\frac{\pi}{C_1}}\, \epsilon^{-\left(\frac{1}{2}+\frac{1}{\lambda}\right)} \tag{4.15}$$

so that

$$\bar{\mathcal{K}}_1(t) \sim \frac{C_4}{\mathcal{J}(\overline{\alpha}, t)^{\frac{1}{2}+\frac{1}{\lambda}}} \tag{4.16}$$

for $\lambda \in (0, 1]$ and positive constants C_4 determined by

$$C_4 = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\lambda}\right)}{\Gamma\left(1 + \frac{1}{\lambda}\right)} \sqrt{-\frac{\pi M_0}{C_1}}. \tag{4.17}$$

Using (4.13) and (4.16) with (3.19) implies

$$u_x(\gamma(\alpha, t), t) \sim \frac{C}{\mathcal{J}(\overline{\alpha}, t)^{\lambda-1}} \left(\frac{\mathcal{J}(\overline{\alpha}, t)}{\mathcal{J}(\alpha, t)} - \frac{C_4}{C_3} \right) \tag{4.18}$$

for $\eta_* - \eta > 0$ small. But ([11])

$$\Gamma(y + 1) = y\,\Gamma(y), \qquad y \in \mathbb{R}^+,$$

so that

$$\frac{C_4}{C_3} = \frac{\Gamma\left(\frac{1}{\lambda}\right)\,\Gamma\left(\frac{1}{\lambda} - \frac{1}{2} + 1\right)}{\Gamma\left(\frac{1}{\lambda} + 1\right)\,\Gamma\left(\frac{1}{\lambda} - \frac{1}{2}\right)} = 1 - \frac{\lambda}{2} \in [1/2, 1) \tag{4.19}$$

for $\lambda \in (0, 1]$. Then, by (4.18), (2.5)i) and the definition of M_0

$$\begin{aligned} M(t) &\to 0^+, \qquad \alpha = \overline{\alpha}, \\ u_x(\gamma(\alpha, t), t) &\to 0^-, \qquad \alpha \neq \overline{\alpha} \end{aligned} \tag{4.20}$$

as $\eta \uparrow \eta_*$ for all $\lambda \in (0, 1)$. For the threshold parameter $\lambda_* = 1$, we keep track of the positive constant C prior to (4.18) and find that, for $\alpha = \overline{\alpha}$,

$$M(t) \to -\frac{u_0'''(\overline{\alpha})}{(2\pi)^2} > 0 \tag{4.21}$$

as $\eta \uparrow \frac{1}{M_0}$, whereas

$$u_x(\gamma(\alpha, t), t) \to \frac{u_0'''(\overline{\alpha})}{(2\pi)^2} < 0 \tag{4.22}$$

for $\alpha \neq \overline{\alpha}$. Finally, from (3.17)

$$dt = \bar{\mathcal{K}}_0(t)^{2\lambda} d\eta, \tag{4.23}$$

then (4.13) implies

$$t_* - t \sim C \int_\eta^{\eta_*} (1 - \lambda\mu M_0)^{\lambda-2} d\mu. \tag{4.24}$$

As a result, $t_* = +\infty$ for all $\lambda \in (0,1]$. See §5 for examples.

Two-sided, everywhere blow-up for $\lambda \in (1,+\infty)$.

For $\lambda \in (1,+\infty)\backslash\{2\}$, set $b = \frac{1}{\lambda}$ in lemma 4.5 to obtain

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha-\overline{\alpha})^2)^{\frac{1}{\lambda}}} = 2s\epsilon^{-\frac{1}{\lambda}}\, {}_2F_1\left[\frac{1}{2},\frac{1}{\lambda};\frac{3}{2};\frac{s^2C_1}{\epsilon}\right] \tag{4.25}$$

where the above series is defined by (4.2) as long as $\epsilon \geq -C_1 \geq -s^2C_1 > 0$, namely $-1 \leq \frac{s^2C_1}{\epsilon} < 0$. However, we are ultimately interested in the behaviour of (4.25) for $\epsilon > 0$ arbitrarily small, so that, eventually $\frac{s^2C_1}{\epsilon} < -1$. To achieve this transition of the series argument across -1 in a well-defined, continuous fashion, we use proposition 4.3 which provides us with the analytic continuation of the series in (4.25) from argument values inside the unit circle, in particular for the interval $-1 \leq \frac{s^2C_1}{\epsilon} < 0$, to those found outside and thus for $\frac{s^2C_1}{\epsilon} < -1$. Consequently, for ϵ small enough, so that $-s^2C_1 > \epsilon > 0$, proposition 4.3 implies

$$2s\epsilon^{-\frac{1}{\lambda}}\, {}_2F_1\left[\frac{1}{2},\frac{1}{\lambda};\frac{3}{2};\frac{s^2C_1}{\epsilon}\right] = C\,\Gamma\left(\frac{1}{\lambda}-\frac{1}{2}\right)\epsilon^{\frac{1}{2}-\frac{1}{\lambda}} + \frac{C}{\lambda-2} + \psi(\epsilon) \tag{4.26}$$

for $\psi(\epsilon) = o(1)$ as $\epsilon \to 0$ and positive constant C which may depend on λ and can be obtained explicitly from (4.4). Then, substituting $\epsilon = \frac{1}{\lambda\eta} - M_0$ into (4.26) and using (4.9) along with (4.25), yields

$$\bar{\mathcal{K}}_0(t) \sim \begin{cases} C_3\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}-\frac{1}{\lambda}}, & \lambda \in (1,2), \\ C, & \lambda \in (2,+\infty) \end{cases} \tag{4.27}$$

for $\eta_* - \eta > 0$ small and positive constants C_3 given by (4.14) for $\lambda \in (1,2)$. Similarly, by following an identical argument, with $b = 1 + \frac{1}{\lambda}$ instead, we find that estimate (4.16), derived initially for $\lambda \in (0,1]$, holds for $\lambda \in (1,+\infty)$ as well. First suppose $\lambda \in (1,2)$, then (3.19), (4.16) and (4.27)i) imply estimate (4.18). However, by (4.19) we now have

$$\frac{C_4}{C_3} = 1 - \frac{\lambda}{2} \in (0,1/2)$$

for $\lambda \in (1,2)$. As a result, setting $\alpha = \overline{\alpha}$ in (4.18), we obtain

$$M(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)^{\lambda-1}} \to +\infty \tag{4.28}$$

as $\eta \uparrow \eta_*$. On the other hand, if $\alpha \neq \overline{\alpha}$, the definition of M_0 gives

$$u_x(\gamma(\alpha,t),t) \sim -\frac{C}{\mathcal{J}(\overline{\alpha},t)^{\lambda-1}} \to -\infty. \tag{4.29}$$

The existence of a finite $t_* > 0$ follows from (4.23) and (4.27)i), which imply

$$t_* - t \sim C(\eta_* - \eta)^{\lambda-1}. \tag{4.30}$$

For $\lambda \in (2,+\infty)$, we use (3.19), (4.16) and (4.27)ii) to get

$$u_x(\gamma(\alpha,t),t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)}\left(\frac{\mathcal{J}(\overline{\alpha},t)}{\mathcal{J}(\alpha,t)} - C\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}-\frac{1}{\lambda}}\right). \tag{4.31}$$

Then, setting $\alpha = \overline{\alpha}$ in (4.31), we obtain

$$M(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)} \to +\infty \tag{4.32}$$

as $\eta \uparrow \eta_*$. Similarly, for $\alpha \neq \overline{\alpha}$,

$$u_x(\gamma(\alpha,t),t) \sim -\frac{C}{\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}+\frac{1}{\lambda}}} \to -\infty. \tag{4.33}$$

A finite blow-up time $t_* > 0$ follows from (4.23) and (4.27)ii), which yield

$$t_* - t \sim C(\eta_* - \eta).$$

For the case $\lambda = 2$ and $\eta_* - \eta = \frac{1}{2M_0} - \eta > 0$ small, we have

$$\bar{\mathcal{K}}_0(t) \sim -C \ln\left(\mathcal{J}(\overline{\alpha},t)\right), \qquad \bar{\mathcal{K}}_1(t) \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)}. \tag{4.34}$$

Two-sided blow-up for $\lambda = 2$ then follows from (3.19), (4.23) and (4.34). Finally, the behaviour of the jacobian in (4.8) is deduced from (3.15) and the estimates (4.13), (4.27) and (4.34). See §5.1 for examples. □

Remark 4.35. Several methods were used in [4] to show that there are blow-up solutions for $\lambda = 1$ under Dirichlet boundary conditions. We remark that these do not conflict with our global result in part 1 of Theorem 4.7 as long as the data is smooth and, under certain circumstances, its local behaviour near the endpoints $\alpha = \{0, 1\}$ allows for a smooth, periodic extension of u_0' to all $\alpha \in \mathbb{R}$. Further details on this will be given in future work. See also §4.2.2 where a particular choice of $u_0''(\alpha) \in PC_{\mathbb{R}}(0,1)$ leads to finite-time blow-up for all $\lambda \in (1/2, +\infty)$.

4.1.2. Blow-up for $\lambda \in (-\infty, -1)$.
Theorem 4.36 below shows the existence of mean-zero, smooth data for which solutions undergo a two-sided, everywhere blow-up in finite-time for $\lambda \in (-\infty, -2]$, whereas, if $\lambda \in (-2, -1)$, only the minimum diverges.

Theorem 4.36. *Consider the initial boundary value problem (1.1)-(1.2). There exist smooth, mean-zero initial data such that:*

1. *For $\lambda \in (-\infty, -2]$, there is a finite $t_* > 0$ such that both the maximum $M(t)$ and the minimum $m(t)$ diverge to $+\infty$ and respectively to $-\infty$ as $t \uparrow t_*$. In addition, $\lim_{t\uparrow t_*} u_x(\gamma(\alpha,t),t) = +\infty$ for $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$ (two-sided, everywhere blow-up).*
2. *For $\lambda \in (-2, -1)$, there exists a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up).*

Finally, for $\lambda \in (-\infty, -1)$ and t_* as above, the jacobian (3.15) satisfies

$$\lim_{t\uparrow t_*} \gamma_\alpha(\alpha,t) = \begin{cases} 0, & \alpha = \underline{\alpha}_j, \\ C, & \alpha \neq \underline{\alpha}_j \end{cases} \tag{4.37}$$

where the positive constants C depend on the choice of λ and $\alpha \neq \underline{\alpha}_j$.

Proof. For $\lambda \in (-\infty, -1)$, smoothness of u_0' implies that

$$\bar{\mathcal{K}}_0(t) = \int_0^1 \mathcal{J}(\alpha,t)^{\frac{1}{|\lambda|}}\, d\alpha$$

remains finite (and positive) for all $\eta \in [0, \eta_*)$, $\eta_* = \frac{1}{\lambda m_0}$. Also, $\bar{\mathcal{K}}_0(t)$ has a finite, positive limit as $\eta \uparrow \eta_*$. Indeed, suppose there is an earliest $t_1 > 0$ such that $\eta_1 = \eta(t_1) > 0$ and

$$\bar{\mathcal{K}}_0(t_1) = \int_0^1 \left(1 - \lambda\eta_1 u_0'(\alpha)\right)^{\frac{1}{|\lambda|}} d\alpha = 0. \tag{4.38}$$

Since $\int_0^1 (1-u_0'/m_0)^{\frac{1}{|\lambda|}}\,d\alpha > 0$, then $\eta_1 \neq \eta_*$. Also, by periodicity of u_0, there are $[0,1] \ni \alpha_1 \neq \underline{\alpha}_j$ where $(1-\lambda\eta_1 u_0'(\alpha_1))^{\frac{1}{|\lambda|}} = 1$, and so, (4.38) implies the existence of at least one $\alpha' \neq \underline{\alpha}_j$ where $u_0'(\alpha') = \frac{1}{\lambda\eta_1}$. But $u_0' \geq m_0$ and $\eta_* = \frac{1}{\lambda m_0}$, then

$$\eta_* < \eta_1. \tag{4.39}$$

In addition, (4.39) and $m_0 \leq u_0' \leq M_0$ yield

$$0 < \int_0^1 \frac{d\alpha}{(1-u_0'/m_0)^{\frac{1}{\lambda}}} \leq \bar{\mathcal{K}}_0(t) \leq 1, \qquad 0 \leq \eta \leq \eta_*. \tag{4.40}$$

Next, for $\lambda \in (-\infty,-1)$, we estimate $\bar{\mathcal{K}}_1(t) = \int_0^1 \mathcal{J}(\alpha,t)^{\frac{1}{|\lambda|}-1} d\alpha$ as $\eta \uparrow \eta_*$ by following an argument similar to that of Theorem 4.7. For simplicity, suppose m_0 occurs at a single $\underline{\alpha} \in (0,1)$. We consider the case where, near $\underline{\alpha}$, $u_0'(\alpha)$ has non-vanishing second order derivative, so that, locally $u_0'(\alpha) \sim m_0 + C_2(\alpha-\underline{\alpha})^2$ for $0 \leq |\alpha-\underline{\alpha}| \leq r$, $0 < r \leq 1$ and $C_2 = u_0'''(\underline{\alpha})/2 > 0$. Then, for $\epsilon > 0$

$$\epsilon + u_0'(\alpha) - m_0 \sim \epsilon + C_2(\alpha-\underline{\alpha})^2. \tag{4.41}$$

Given $\lambda \in (-\infty,-1)$, set $b = 1+\frac{1}{\lambda}$ in lemma (4.5) to find

$$\int_{\underline{\alpha}-r}^{\underline{\alpha}+r} \frac{d\alpha}{(\epsilon + C_2(\alpha-\underline{\alpha})^2)^{1+\frac{1}{\lambda}}} = \frac{2r}{\epsilon^{1+\frac{1}{\lambda}}}\, {}_2F_1\left[\frac{1}{2}, 1+\frac{1}{\lambda}; \frac{3}{2}; -\frac{r^2C_2}{\epsilon}\right] \tag{4.42}$$

for $\epsilon \geq C_2 \geq r^2C_2$, i.e. $-1 \leq -\frac{r^2C_2}{\epsilon} < 0$, and $\lambda \in (-\infty,-1)\backslash\{-2\}$[4]. Then, as we let $\epsilon > 0$ become small enough, so that eventually $-\frac{r^2C_2}{\epsilon} < -1$, Proposition 4.3 implies

$$\frac{2r}{\epsilon^{1+\frac{1}{\lambda}}}\, {}_2F_1\left[\frac{1}{2}, 1+\frac{1}{\lambda}; \frac{3}{2}; -\frac{r^2C_2}{\epsilon}\right] = \frac{C}{\lambda+2} + \frac{C\,\Gamma\left(\frac{1}{2}+\frac{1}{\lambda}\right)}{\epsilon^{\frac{1}{2}+\frac{1}{\lambda}}} + \xi(\epsilon) \tag{4.43}$$

for $\xi(\epsilon) = o(1)$ as $\epsilon \to 0$ and positive constants C which may depend on the choice of λ and can be obtained explicitly from (4.4). Using (4.43) on (4.42), along with (4.41) and the substitution $\epsilon = m_0 - \frac{1}{\lambda\eta}$, yields

$$\bar{\mathcal{K}}_1(t) \sim \begin{cases} C, & \lambda \in (-2,-1), \\ C_5 \mathcal{J}(\underline{\alpha},t)^{-\left(\frac{1}{2}+\frac{1}{\lambda}\right)}, & \lambda \in (-\infty,-2) \end{cases} \tag{4.44}$$

for $\eta_* - \eta > 0$ small, $\eta_* = \frac{1}{\lambda m_0}$, $\mathcal{J}(\underline{\alpha},t) = 1-\lambda\eta(t)m_0$ and

$$C_5 = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{\lambda}\right)}{\Gamma\left(1+\frac{1}{\lambda}\right)}\sqrt{-\frac{\pi m_0}{C_2}} > 0, \qquad \lambda \in (-\infty,-2). \tag{4.45}$$

Setting $\alpha = \underline{\alpha}$ in (3.19) and using (2.5)ii), (4.40) and (4.44), implies

$$m(t) \sim -\frac{C}{\mathcal{J}(\underline{\alpha},t)} \to -\infty \tag{4.46}$$

as $\eta \uparrow \eta_*$ for all $\lambda \in (-\infty,-1)\backslash\{-2\}$. On the other hand, using (3.19), (4.40), (4.44) and the definition of m_0, we see that, for $\alpha \neq \underline{\alpha}$,

$$\begin{cases} |u_x(\gamma(\alpha,t),t)| < +\infty, & \lambda \in (-2,-1), \\ u_x(\gamma(\alpha,t),t) \sim C\mathcal{J}(\underline{\alpha},t)^{-\left(\frac{1}{2}+\frac{1}{\lambda}\right)} \to +\infty, & \lambda \in (-\infty,-2) \end{cases} \tag{4.47}$$

[4]The case $\lambda = -2$ is treated separately.

as $\eta \uparrow \eta_*$. A one-sided, discrete blow-up for $\lambda \in (-2,-1)$ follows from (4.46) and (4.47)i), whereas a two-sided, everywhere blow-up for $\lambda \in (-\infty,-2)$ results from (4.46) and (4.47)ii). The existence of a finite $t_* > 0$ follows from (3.17) and (4.40) as $\eta \uparrow \eta_*$. Particularly, we have the lower bound

$$\eta_* \leq t_* < +\infty. \tag{4.48}$$

The case $\lambda = -2$ can be treated directly. We find

$$\bar{\mathcal{K}}_1(t) \sim -C \ln\left(\mathcal{J}(\underline{\alpha}, t)\right) \tag{4.49}$$

for $\eta_* - \eta > 0$ small. A two-sided blow-up then follows as above. Finally, (4.37) is deduced from (3.15) and (4.40). See §5.1 for examples. □

4.1.3. One-sided, discrete blow-up for $\lambda \in [-1,0)$.
Observe that (3.2) and $\gamma_\alpha(\alpha,0) = 1$ imply the existence of a time interval $[0,t_*)$ where

$$\gamma_\alpha(\alpha,t) = \exp\left(\int_0^t u_x(\gamma(\alpha,s),s)\,ds\right) > 0 \tag{4.50}$$

for all $\alpha \in [0,1]$ and $0 < t_* \leq +\infty$. Then, given $\lambda \in [-1,0)$, (3.4) implies that

$$(\gamma_\alpha^{-\lambda}(\alpha,t))^{\cdot\cdot} \leq 0.$$

But $(\gamma_\alpha^{-\lambda})^{\cdot}|_{t=0} = -\lambda u_0'$, thus integrating the above twice in time gives

$$\gamma_\alpha(\alpha,t)^{-\lambda} \leq 1 - \lambda t\, u_0'(\alpha).$$

Provided there is $\underline{\alpha} \in [0,1]$ such that $\inf_{\alpha\in[0,1]} u_0'(\alpha) = u_0'(\underline{\alpha}) < 0$, define

$$t_* = (\lambda u_0'(\underline{\alpha}))^{-1},$$

then $\gamma_\alpha(t,\underline{\alpha}) \downarrow 0$ as $t \uparrow t_*$ for any $\lambda \in [-1,0)$. This along with (2.5)ii) and (4.50) implies

$$\lim_{t\uparrow t_*} \int_0^t u_x(\gamma(\underline{\alpha},s),s)\,ds = \lim_{t\uparrow t_*} \int_0^t m(s)ds = -\infty. \tag{4.51}$$

More precise blow-up properties are now studied via formula (3.19). Theorem 4.52 below will extend the one-sided, discrete blow-up found in Theorem 4.36 for parameters $\lambda \in (-2,-1)$ to all $\lambda \in (-2,0)$.

Theorem 4.52. *Consider the initial boundary value problem (1.1)-(1.2) with arbitrary smooth, mean-zero initial data. For every $\lambda \in [-1,0)$, there exists a finite $t_* > 0$ such that only the minimum diverges, $m(t) \to -\infty$, as $t \uparrow t_*$ (one-sided, discrete blow-up). Also, the jacobian (3.15) satisfies*

$$\lim_{t\uparrow t_*} \gamma_\alpha(\alpha,t) = \begin{cases} 0, & \alpha = \underline{\alpha}_j, \\ C, & \alpha \neq \underline{\alpha}_j \end{cases} \tag{4.53}$$

where the positive constants C depend on the choice of λ and $\alpha \neq \underline{\alpha}_j$.

Proof. Since u_0' is smooth and $\lambda \in [-1,0)$, both integrals $\bar{\mathcal{K}}_i(t)$, $i = 0,1$ remain finite (and positive) for all $\eta \in [0,\eta_*)$, $\eta_* = \frac{1}{\lambda m_0}$. Also, $\bar{\mathcal{K}}_0(t)$ does not vanish as $\eta \uparrow \eta_*$. In fact

$$1 \leq \bar{\mathcal{K}}_0(t) \leq \left(1 - \frac{M_0}{m_0}\right)^{\frac{1}{|\lambda|}} \tag{4.54}$$

for all $\eta \in [0, \eta_*]$. Indeed, notice that $\dot{\bar{\mathcal{K}}}_0(0) = 0$ and

$$\ddot{\bar{\mathcal{K}}}_0(t) = \left((1+\lambda) \int_0^1 \frac{u_0'(\alpha)^2 d\alpha}{\mathcal{J}(\alpha,t)^{2+\frac{1}{\lambda}}} - 2\lambda \left(\int_0^1 \frac{u_0'(\alpha)\, d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}} \right)^2 \right) \bar{\mathcal{K}}_0(t)^{-4\lambda} > 0$$

for $\lambda \in [-1, 0)$ and $\eta \in (0, \eta_*)$. This implies

$$\dot{\bar{\mathcal{K}}}_0(t) = \bar{\mathcal{K}}_0(t)^{-2\lambda} \int_0^1 \frac{u_0'(\alpha) d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}} > 0. \tag{4.55}$$

Then, using (4.55), $\bar{\mathcal{K}}_0(0) = 1$ and $m_0 \le u_0'(\alpha) \le M_0$ yield (4.54). Similarly, one can show that

$$1 \le \bar{\mathcal{K}}_1(t) \le \left(\frac{m_0}{m_0 - M_0} \right)^{1+\frac{1}{\lambda}}. \tag{4.56}$$

Consequently, (2.5)ii), (3.19), (4.54) and (4.56) imply that

$$m(t) = u_x(\gamma(\underline{\alpha}_j, t), t) \to -\infty$$

as $\eta \uparrow \eta_*$. On the other hand, by (4.54), (4.56) and the definition of m_0, we find that $u_x(\gamma(\alpha, t), t)$ remains bounded for all $\alpha \neq \underline{\alpha}_j$ as $\eta \uparrow \eta_*$. The existence of a finite blow-up time $t_* > 0$ is guaranteed by (3.17) and (4.54). Although t_* can be computed explicitly from (2.1), (4.54) provides the simple estimate[5]

$$\eta_* \left(1 - \frac{M_0}{m_0} \right)^{-2} \le t_* \le \eta_*. \tag{4.57}$$

Also, since the maximum $M(t)$ remains finite as $t \uparrow t_*$, setting $\alpha = \overline{\alpha}_i$ in (3.19) and using (2.5)i) and (3.1) gives $\dot{M}(t) < \lambda (M(t))^2 < 0$, which implies

$$0 < M(t) \le M_0$$

for all $t \in [0, t_*]$ and $\lambda \in [-1, 0)$. Finally, (4.53) follows directly from (3.15), (4.54) and the definition of m_0. See §5.1 for examples. □

4.1.4. Further L^p Regularity.

In this section, we prove Theorem 2.11. In particular, we will see how the two-sided, everywhere blow-up (or one-sided, discrete blow-up) found in theorems 4.7, 4.36 and 4.52, can be associated with stronger (or weaker) L^p regularity. Before proving the Theorem, we derive basic upper and lower bounds for $\|u_x\|_p$, $p \in [1, +\infty)$, as well as write down explicit formulas for the energy function $E(t) = \|u_x\|_2^2$ and derivative $\dot{E}(t)$, and estimate the blow-up rates of relevant time-dependent integrals. From (3.15) and (3.19),

$$|u_x(\gamma(\alpha,t),t)|^p \, \gamma_\alpha(\alpha,t) = \frac{|f(\alpha,t)|^p}{|\lambda\eta(t)|^p \, \bar{\mathcal{K}}_0(t)^{1+2\lambda p}} \tag{4.58}$$

for $t \in [0, t_*)$, $p \in [1, +\infty)$, $\lambda \neq 0$ and

$$f(\alpha,t) = \frac{1}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t) \mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}.$$

[5] Which we may contrast to (4.48). Notice that (2.1) implies that the two cases coincide ($t_* = \eta_*$) in the case of Burgers' equation $\lambda = -1$.

Integrating (4.58) in α and using periodicity then gives

$$\|u_x(x,t)\|_p^p = \frac{1}{|\lambda\eta(t)|^p\,\bar{\mathcal{K}}_0(t)^{1+2\lambda p}}\int_0^1 |f(\alpha,t)|^p\,d\alpha. \tag{4.59}$$

In particular, setting $p=2$ yields the following formula for the energy $E(t)$:

$$E(t) = \left(\lambda\eta(t)\bar{\mathcal{K}}_0(t)^{1+2\lambda}\right)^{-2}\left(\bar{\mathcal{K}}_0(t)\bar{\mathcal{K}}_2(t) - \bar{\mathcal{K}}_1(t)^2\right). \tag{4.60}$$

Furthermore, multiplying (1.1)i) by u_x, integrating by parts and using (1.2), (3.15) and (3.19) gives, after some simplification,

$$\begin{aligned}\dot{E}(t) &= (1+2\lambda)\int_0^1 u_x(x,t)^3 dx \\ &= (1+2\lambda)\int_0^1 (u_x(\gamma(\alpha,t),t))^3\gamma_\alpha(\alpha,t)\,d\alpha \\ &= \frac{1+2\lambda}{(\lambda\eta(t))^3}\left[\frac{\bar{\mathcal{K}}_3(t)}{\bar{\mathcal{K}}_1(t)} - \frac{3\bar{\mathcal{K}}_2(t)}{\bar{\mathcal{K}}_0(t)} + 2\left(\frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\right)^2\right]\frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)^{1+6\lambda}}.\end{aligned} \tag{4.61}$$

Now $\bar{\mathcal{K}}_i(t)$, $\mathcal{J}(\alpha,t) > 0$ for $\eta \in [0,\eta_*)$ (i.e. $t \in [0,t_*)$) and $\alpha \in [0,1]$. As a result

$$|f(\alpha,t)|^p \le 2^{p-1}\left(\frac{1}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} + \frac{\bar{\mathcal{K}}_1(t)^p}{\bar{\mathcal{K}}_0(t)^p\,\mathcal{J}(\alpha,t)^{\frac{1}{\lambda}}}\right)$$

which can be used together with (4.58) to obtain, upon integration, the upper bound

$$\|u_x(x,t)\|_p^p \le \frac{2^{p-1}}{|\lambda\eta(t)|^p\,\bar{\mathcal{K}}_0(t)^{1+2\lambda p}}\left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} + \frac{\bar{\mathcal{K}}_1(t)^p}{\bar{\mathcal{K}}_0(t)^{p-1}}\right) \tag{4.62}$$

valid for $t \in [0,t_*)$, $p \in [1,+\infty)$ and $\lambda \neq 0$. For a lower bound, notice that by Jensen's inequality,

$$\int_0^1 |f(\alpha,t)|^p\,d\alpha \ge \left|\int_0^1 f(\alpha,t)d\alpha\right|^p$$

for $p \in [1,+\infty)$. Using the above in (4.59), we find

$$\|u_x(x,t)\|_p \ge \frac{1}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}}\left|\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} - \frac{\bar{\mathcal{K}}_1(t)}{\bar{\mathcal{K}}_0(t)}\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}\right|. \tag{4.63}$$

Although the right-hand side of (4.63) is identically zero for $p=1$, it does allow for the study of L^p regularity of solutions when $p \in (1,+\infty)$[6]. Before proving Theorem 2.11, we need to determine any blow-up rates for the appropriate integrals in (4.60)-(4.63). By following the argument in theorems 4.7 and 4.36, we go through the derivation of estimates for

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}}$$

with $\lambda \in (1,+\infty)$, $p \in [1+\infty)$ and $\eta_* = \frac{1}{\lambda M_0}$, whereas those for

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}}, \qquad \int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}}$$

[6] Also, for $p \in (1,+\infty)$, (4.63) makes sense as $t \downarrow 0$ due to the periodicity of u_0'.

follow similarly and will be simply stated here. For simplicity, assume u_0' attains its maximum value $M_0 > 0$ at a single $\overline{\alpha} \in (0,1)$. As before, we consider the case where, near $\overline{\alpha}$, u_0' has non-vanishing second-order derivative. Accordingly, there is $s \in (0,1]$ such that $u_0'(\alpha) \sim M_0 + C_1(\alpha - \overline{\alpha})^2$ for $0 \leq |\alpha - \overline{\alpha}| \leq s$ and $C_1 = u_0'''(\overline{\alpha})/2 < 0$. Then

$$\epsilon - u_0'(\alpha) + M_0 \sim \epsilon - C_1(\alpha - \overline{\alpha})^2$$

for $\epsilon > 0$. Given $\lambda > 1$ and $p \geq 1$, we let $b = 1 + \frac{1}{\lambda p}$ in lemma 4.5 to obtain

$$\int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - u_0'(\alpha) + M_0)^b} \sim \int_{\overline{\alpha}-s}^{\overline{\alpha}+s} \frac{d\alpha}{(\epsilon - C_1(\alpha-\overline{\alpha})^2)^b} = \frac{2s}{\epsilon^b}\, {}_2F_1\left[\frac{1}{2}, b; \frac{3}{2}; \frac{C_1 s^2}{\epsilon}\right] \tag{4.64}$$

for $\epsilon \geq -C_1 \geq -s^2 C_1 > 0$. Now, if we let $\epsilon > 0$ become small enough, so that eventually $\frac{C_1 s^2}{\epsilon} < -1$, proposition 4.3 implies

$$\frac{2s}{\epsilon^b}\, {}_2F_1\left[\frac{1}{2}, b; \frac{3}{2}; \frac{C_1 s^2}{\epsilon}\right] = \frac{2s}{(1-2b)(-s^2 C_1)^b} + \frac{\Gamma\left(b - \frac{1}{2}\right)}{\Gamma(b)} \sqrt{-\frac{\pi}{C_1}}\, \epsilon^{\frac{1}{2}-b} + \zeta(\epsilon)$$

for $\lambda \neq 2/p$, and $\zeta(\epsilon) = o(1)$ as $\epsilon \to 0$. Using the above on (4.64) yields

$$\int_0^1 \frac{d\alpha}{(\epsilon - u_0'(\alpha) + M_0)^b} \sim \frac{\Gamma(b - 1/2)}{\Gamma(b)} \sqrt{-\frac{\pi}{C_1}}\, \epsilon^{\frac{1}{2}-b} \tag{4.65}$$

for $\epsilon > 0$ small. Then, setting $\epsilon = \frac{1}{\lambda \eta} - M_0$ into (4.65) gives

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} \sim C \mathcal{J}(\overline{\alpha}, t)^{-(\frac{1}{2} + \frac{1}{\lambda p})} \tag{4.66}$$

for $\eta_* - \eta > 0$ small, $\eta_* = \frac{1}{\lambda M_0}$, $p \in [1, +\infty)$ and $\lambda \in (1, +\infty)$[7]. For the other cases and remaining integrals, we follow a similar argument to find

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda p}}} \sim \frac{C}{\mathcal{J}(\underline{\alpha},t)^{\frac{1}{2}+\frac{1}{\lambda p}}}, \qquad \lambda < -\frac{2}{p}, \quad p \in [1, +\infty), \tag{4.67}$$

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{\frac{1}{\lambda p}}} \sim \begin{cases} C, & \lambda > \frac{2}{p},\ p \geq 1 \ \text{ or } \ \lambda \in \mathbb{R}^-, \\ C\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}-\frac{1}{\lambda p}}, & 1 < \lambda < \frac{2}{p}, \quad 1 < p < 2 \end{cases} \tag{4.68}$$

and

$$\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} \sim C, \qquad \frac{2}{1-2p} < \lambda < 0, \ \ p \geq 1 \tag{4.69}$$

where the positive constants C may depend on the choices for λ and p.
Recall from Theorem 4.7 (see also appendix A) that

$$\lim_{t \to +\infty} \|u_x\|_\infty < +\infty, \qquad \lambda \in [0,1]. \tag{4.70}$$

In contrast, Theorem 2.6, which we established in Theorems 4.7, 4.36 and 4.52, showed the existence of a finite $t_* > 0$ such that

$$\lim_{t \uparrow t_*} \|u_x\|_\infty = +\infty, \qquad \lambda \in \mathbb{R} \backslash [0,1]. \tag{4.71}$$

Consequently, $\|u_x\|_p$ exists globally for all $p \in [1, +\infty]$ and $\lambda \in [0,1]$. In the case of (4.71), Theorem 4.72 below further examines the L^p regularity of u_x as t approaches the finite L^∞ blow-up time t_*.

[7] When $\lambda = 2/p$, $b = 3/2$ and (4.66) reduces to (4.34)ii).

Theorem 4.72. *Consider the initial boundary value problem (1.1)-(1.2) and let $t_* > 0$ denote the finite L^∞ blow-up time in Theorem 2.6. There exist smooth, mean-zero initial data such that:*

1. *For $p \in (1, +\infty)$ and $\lambda \in (-\infty, -2/p] \cup (1, +\infty)$, $\lim_{t\uparrow t_*} \|u_x\|_p = +\infty$.*
2. *For $p \in [1, +\infty)$ and $\frac{2}{1-2p} < \lambda < 0$, $\lim_{t\uparrow t_*} \|u_x\|_p < +\infty$.*
3. *The energy $E(t) = \|u_x\|_2^2$ diverges as $t \uparrow t_*$ if $\lambda \in (-\infty, -2/3] \cup (1, +\infty)$ but remains finite for $t \in [0, t_*]$ when $\lambda \in (-2/3, 0)$. Moreover, $\dot{E}(t)$ blows up to $+\infty$ as $t \uparrow t_*$ if $\lambda \in (-\infty, -1/2) \cup (1, +\infty)$ and $\dot{E}(t) \equiv 0$ for $\lambda = -1/2$; whereas, $\lim_{t\uparrow t_*} \dot{E}(t) = -\infty$ when $\lambda \in (-1/2, -2/5]$ but remains bounded, for all $t \in [0, t_*]$, if $\lambda \in (-2/5, 0)$.*

Proof. **Case** λ, $p \in (1, +\infty)$.
First, consider the lower bound (4.63) for $p \in (1, 2)$ and $\lambda \in (1, 2/p)$. Then, $\lambda \in (1, 2)$ so that (4.16), (4.27)i), (4.66) and (4.68)ii) imply

$$\|u_x(x,t)\|_p^p \geq \frac{\left|\int_0^1 f(\alpha,t)d\alpha\right|^p}{|\lambda\eta(t)|^p\, \bar{\mathcal{K}}_0(t)^{1+2\lambda p}} \sim C\mathcal{J}(\overline{\alpha},t)^{\sigma(\lambda,p)}$$

for $\eta_* - \eta > 0$ small and $\sigma(\lambda, p) = \frac{3p}{2} - \frac{1}{2} - \lambda p$. By the above restrictions on λ and p, we see that $\sigma(\lambda, p) < 0$ for

$$\frac{1}{2}\left(3 - \frac{1}{p}\right) < \lambda < \frac{2}{p}, \qquad p \in (1, 5/3).$$

Then, by choosing $p - 1 > 0$ arbitrarily small, $\|u_x\|_p \to +\infty$ as $t \uparrow t_*$ for $\lambda \in (1, 2)$. Next, let $\lambda \in (2, +\infty)$ and $p \in (1, +\infty)$. This means $\lambda > \frac{2}{p}$, and so, (4.16), (4.27)ii), (4.66) and (4.68)i) now yield

$$\|u_x(x,t)\|_p \geq \frac{\left|\int_0^1 f(\alpha,t)d\alpha\right|}{|\lambda\eta(t)|\, \bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}} \sim \frac{C}{\mathcal{J}(\overline{\alpha},t)^{\frac{1}{2}+\frac{1}{\lambda}}} \to +\infty \tag{4.73}$$

as $t \uparrow t_*$. This proves part (1) of the Theorem for $\lambda \in (1, +\infty)$[8].
Case $\lambda \in (-\infty, 0)$ **and** $p \in [1, +\infty)$**.**
For $\lambda \in (-\infty, 0)$, we keep in mind the estimates (4.40), (4.44)i), (4.54) and (4.56) which describe the behaviour of $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$ as $\eta \uparrow \eta_*$. Consider the upper bound (4.62) for $p \in [1, +\infty)$ and $\frac{2}{1-2p} < \lambda < 0$. Then $\lambda \in (-2, 0)$, equation (4.69) and the aforementioned estimates imply

$$\|u_x(x,t)\|_p^p \leq \frac{2^{p-1}}{|\lambda\eta(t)|^p\, \bar{\mathcal{K}}_0(t)^{1+2\lambda p}} \left(\int_0^1 \frac{d\alpha}{\mathcal{J}(\alpha,t)^{p+\frac{1}{\lambda}}} + \frac{\bar{\mathcal{K}}_1(t)^p}{\bar{\mathcal{K}}_0(t)^{p-1}}\right) \to C$$

as $t \uparrow t_*$. Here, $C \in \mathbb{R}^+$ depends on the choice of λ and p. By the above, we conclude that

$$\lim_{t\uparrow t_*} \|u_x(x,t)\|_p < +\infty$$

for $\frac{2}{1-2p} < \lambda < 0$ and $p \in [1, +\infty)$. Now, consider the lower bound (4.63) with $p \in (1, +\infty)$ and $-2 < \lambda < -\frac{2}{p} < \frac{2}{1-2p}$. Then, by (4.67), (4.68)i) and corresponding

[8] If $\lambda = 2$, $\lambda > \frac{2}{p}$ for $p > 1$ and result follows from (4.34), (4.63), (4.66) and (4.68)i).

estimates on $\bar{\mathcal{K}}_i(t)$, $i = 0, 1$, we find that

$$\|u_x(x,t)\|_p \geq \frac{\left|\int_0^1 f(\alpha,t)d\alpha\right|}{|\lambda\eta(t)|\,\bar{\mathcal{K}}_0(t)^{2\lambda+\frac{1}{p}}} \sim C\mathcal{J}(\underline{\alpha},t)^{-\left(\frac{1}{2}+\frac{1}{\lambda}\right)} \tag{4.74}$$

for $\eta_* - \eta > 0$ small. Therefore,

$$\lim_{t\uparrow t_*} \|u_x(x,t)\|_p = +\infty \tag{4.75}$$

for $p \in (1,+\infty)$ and $\lambda \in (-2,-2/p]$[9]. Finally, let $\lambda \in (-\infty,-2)$ and $p \in (1,+\infty)$. Then $\lambda < -\frac{2}{p}$ and it is easy to check that (4.74), with different constants $C > 0$, also holds. As a result, (4.75) follows for $p > 1$ and $\lambda \in (-\infty,-2]$[10]. Since we already established that $u_x \in L^\infty$ for all time when $\lambda \in [0,1]$ (see Theorem 4.7), this concludes the proof of parts (1) and (2) of the Theorem.

For part (3), notice that when $p = 2$, parts (1) and (2), as well as Theorem 4.7 imply that, as $t \uparrow t_*$, both $E(t) = \|u_x\|_2^2$ and $\dot{E}(t)$ diverge to $+\infty$ for $\lambda \in (-\infty,-1] \cup (1,+\infty)$ while $E(t)$ remains finite if $\lambda \in (-2/3,1]$. Therefore we still have to establish the behaviour of $E(t)$ when $\lambda \in (-1,-2/3]$ and $\dot{E}(t)$ for $\lambda \in (-1,0)\backslash\{-1/2\}$. From (4.54), (4.56) and (4.60), we see that, as $t \uparrow t_*$, any blow-up in $E(t)$ for $\lambda \in (-1,-2/3]$ must come from the $\bar{\mathcal{K}}_2(t)$ term. Using proposition 4.3 and lemma 4.5, we estimate[11]

$$\bar{\mathcal{K}}_2(t) \sim \begin{cases} C\mathcal{J}(\underline{\alpha},t)^{-\left(\frac{3}{2}+\frac{1}{\lambda}\right)}, & \lambda \in (-1,-2/3), \\ -C\log\left(\mathcal{J}(\underline{\alpha},t)\right), & \lambda = -2/3, \\ C, & \lambda \in (-2/3,0) \end{cases} \tag{4.76}$$

for $\eta_* - \eta > 0$ small. Then, (4.54), (4.56), and (4.60) imply that, as $t \uparrow t_*$, both $E(t)$ and $\dot{E}(t)$ blow-up to $+\infty$ for $\lambda \in (-1,-2/3]$. Now, from (4.61)i), we have that

$$\left|\dot{E}(t)\right| \leq |1+2\lambda|\,\|u_x\|_3^3 \tag{4.77}$$

so that Theorem 4.7 implies that $\dot{E}(t)$ remains finite for all time if $\lambda \in [0,1]$. Also, since $3+\frac{1}{\lambda} \leq 0$ for all $\lambda \in [-1/3,0)$, we use (4.54), (4.56) and (4.76)iii) on (4.61)iii) to conclude that

$$\lim_{t\uparrow t_*} \left|\dot{E}(t)\right| < +\infty$$

for $\lambda \in [-1/3,0)$ as well. Moreover, by part (2), $\lim_{t\uparrow t_*} \|u_x\|_3 < +\infty$ for $\lambda \in (-2/5,0)$. Then, (4.77) implies that $\dot{E}(t)$ also remains finite for $\lambda \in (-2/5,-1/3)$. Lastly, estimating $\bar{\mathcal{K}}_3(t)$ yields

$$\bar{\mathcal{K}}_3(t) \sim \begin{cases} C\mathcal{J}(\underline{\alpha},t)^{-\left(\frac{5}{2}+\frac{1}{\lambda}\right)}, & \lambda \in (-2/3,-2/5), \\ -C\log\left(\mathcal{J}(\underline{\alpha},t)\right), & \lambda = -2/5. \end{cases} \tag{4.78}$$

As a result, (4.54), (4.56), (4.76)iii) and (4.61)iii) imply that

$$\lim_{t\uparrow t_*} \dot{E}(t) = +\infty$$

for $\lambda \in (-2/3,-1/2)$ but

$$\dot{E}(t) \to -\infty$$

[9]For the case $\lambda = -\frac{2}{p}$ with $p \in (1,+\infty)$, we simply use (4.49) instead of (4.67).

[10]If $\lambda = -2$, $\lambda < -\frac{2}{p}$ for $p > 1$. Result follows as above with (4.49) instead of (4.67).

[11]Under the usual assumption $u_0'''(\underline{\alpha}) \neq 0$.

when $\lambda \in (-1/2, -2/5]$. This concludes the proof of the Theorem. We refer the reader to table 1 in §2 for a summary of the above results. □

Notice that Theorems 2.6, 4.72 and inequality (4.77) yield a complete description of the L^3 regularity for u_x: if $\lambda \in [0,1]$, $\lim_{t\to+\infty} \|u_x\|_3 = C$ where $C \in \mathbb{R}^+$ for $\lambda = 1$ but $C = 0$ when $\lambda \in (0,1)$, whereas, for $t_* > 0$ the finite L^∞ blow-up time for u_x in Theorem 2.6,

$$\lim_{t\uparrow t_*} \|u_x(x,t)\|_3 = \begin{cases} +\infty, & \lambda \in (-\infty, -2/5] \cup (1, +\infty), \\ C, & \lambda \in (-2/5, 0) \end{cases} \tag{4.79}$$

where the positive constants C depend on the choice of $\lambda \in (-2/5, 0)$.

Remark 4.80. Theorem 4.72 implies that for every $p > 1$, L^p blow-up occurs for u_x if $\lambda \in \mathbb{R}\backslash(-2, 1]$, whereas for $\lambda \in (-2, 0)$, u_x remains in L^1 but blows up in particular, smaller L^p spaces. This suggests a weaker type of blow-up for the latter which certainly agrees with our L^∞ results where a "stronger", two-sided, everywhere blow-up takes place for $\lambda \in \mathbb{R}\backslash(-2, 1]$, but a "weaker", one-sided, discrete blow-up occurs when $\lambda \in (-2, 0)$.

Remark 4.81. For $V(t) = \int_0^1 u_x^3 dx$, the authors in [17] derived a finite upper bound

$$T^* = \left(\frac{3}{|1 + 2\lambda|\, E(0)} \right)^{\frac{1}{2}} \tag{4.82}$$

for the blow-up time of $E(t)$ for $\lambda < -1/2$ and

$$V(0) < 0, \qquad \frac{|1 + 2\lambda|}{2} V(0)^2 \geq \frac{2}{3} E(0)^3. \tag{4.83}$$

If (4.83)i) holds but we reverse (4.83)ii), then they proved that $\dot{E}(t)$ blows up instead. Now, from Theorem 4.72(3) we have that, in particular for $\lambda \in (-2/3, -1/2)$, $E(t)$ remains bounded for all $t \in [0, t_*]$ but $\dot{E}(t) \to +\infty$ as $t \uparrow t_*$. Here, $t_* > 0$ denotes the finite L^∞ blow-up time for u_x (see Theorem 4.52) and satisfies (4.57). Therefore, further discussion is required to clarify the apparent discrepancy between the two results for $\lambda \in (-2/3, -1/2)$ and u_0' satisfying both conditions in (4.83). Our claim is that for these values of λ,

$$t_* < T^*. \tag{4.84}$$

Specifically, $E(t)$ remains finite for all $t \in [0, t_*] \subset [0, T^*]$, while $\dot{E}(t) \to +\infty$ as $t \uparrow t_*$. From (4.61)i) and (4.83)ii), we have that $\frac{\dot{E}(0)^2}{2|1+2\lambda|} \geq \frac{2}{3} E(0)^3$, or

$$\frac{1}{(|1 + 2\lambda|\, E(0))^3} \geq \frac{4}{3(|1 + 2\lambda|\, \dot{E}(0))^2}.$$

As a result, (4.82) yields

$$T^* \geq \left(\frac{6}{|1 + 2\lambda|\, \dot{E}(0)} \right)^{\frac{1}{3}} \tag{4.85}$$

where we used $\dot{E}(0) > 0$; a consequence of (4.61)i), (4.83)i) and $\lambda \in (-2/3, -1/2)$. Now, for instance, suppose $0 < M_0 \leq |m_0|$ [12]. Then

$$-V(0) = \left| \int_0^1 u_0'(x)^3 dx \right| \leq \max_{x\in[0,1]} \left| u_0'(x) \right|^3 = |m_0|^3, \tag{4.86}$$

[12] A natural case to consider given (4.83)i).

which we use on (4.61)i) to obtain $0 < \dot{E}(0) \leq |1+2\lambda|\,|m_0|^3$, or equivalently

$$\frac{6}{|1+2\lambda|\,\dot{E}(0)} \geq \frac{6}{|1+2\lambda|^2\,|m_0|^3}. \tag{4.87}$$

Consequently, (4.57), (4.85) and (4.87) yield

$$T^* \geq \left(\frac{6}{|1+2\lambda|^2\,|m_0|^3}\right)^{\frac{1}{3}} > \frac{1}{|1+2\lambda|^{\frac{2}{3}}\,|m_0|} > \frac{1}{|\lambda|\,|m_0|} = \eta_* \geq t_* \tag{4.88}$$

for $\lambda \in (-2/3, -1/2)$. If $\lambda \leq -2/3$, both results concerning L^2 blow-up of u_x coincide. Furthermore, in [5] the authors derived a finite upper bound $T_* = \frac{3}{(1+3\lambda)}V(0)^{-\frac{1}{3}}$ for the blow-up time of $V(t)$ to negative infinity valid as long as $V(0) < 0$ and $\lambda < -1/3$. Clearly, T_* also serves as an upper bound for the breakdown of $\|u_x\|_3$ for $\lambda < -1/3$, or $\dot{E}(t) = (1+2\lambda)V((t)$ if $\lambda \in (-\infty, -1/3)\backslash\{-1/2\}$. However, (4.79) and Theorem 4.72(1) prove the existence of a finite $t_* > 0$ such that, particularly for $\lambda \in (-2/5, -1/3]$, $\|u_x\|_3$ remains finite for $t \in [0, t_*]$ while $\lim_{t\uparrow t_*}\|u_x\|_6 = +\infty$. This in turn implies the local boundedness of $\dot{E}(t)$ for $t \in [0, t_*]$ and $\lambda \in (-2/5, -1/3]$. Similar to the previous case, we claim that $t_* < T_*$. Here, once again, we consider the case $0 < M_0 \leq |m_0|$. Accordingly, (4.57) and (4.86) imply

$$T_* = \frac{3}{(1+3\lambda)V(0)^{\frac{1}{3}}} \geq \frac{3}{|1+3\lambda|\,|m_0|} > \frac{1}{|\lambda|\,|m_0|} = \eta_* \geq t_*.$$

For the remaining values $\lambda \leq -2/5$, both our results and those established in [5] regarding blow-up of $V(t)$ agree. A simple example is given by $u_0'(x) = \sin(2\pi x) + \cos(4\pi x)$ for which $V(0) = -3/4$, $E(0) = 1$, $m_0 = -2$ and $M_0 \sim 1.125$. Then, for $\lambda = -3/5 \in (-2/3, -1/2)$, we have $T^* = \sqrt{15} > \eta_* = 5/6 \geq t_* \geq 0.34$, whereas, if $\lambda = -7/20 \in (-2/5, -1/3)$, $T_* = 20(6)^{2/3} > 10/7 = \eta_* \geq t_* \geq 0.59$.

4.2. Piecewise Constant and Piecewise Linear Initial Data

In the previous section, we took smooth data u_0' which attained its extreme values $M_0 > 0 > m_0$ at finitely many points $\overline{\alpha}_i$ and $\underline{\alpha}_j \in [0,1]$, respectively, with u_0' having, relative to the sign of λ, quadratic local behaviour near these locations. In this section, two other classes of data are considered which violate these assumptions. In §4.2.1, L^p regularity of solutions is examined for $u_0'(\alpha) \in PC_{\mathbb{R}}(0,1)$, the class of mean-zero, piecewise constant functions. Subsequently, the case $u_0''(\alpha) \in PC_{\mathbb{R}}(0,1)$ is examined via a simple example in §4.2.2.

4.2.1. $n-$phase Piecewise Constant $u_0'(x)$.

Let $\chi_i(\alpha)$, $i = 1, ..., n$ denote the characteristic function for the intervals $\Omega_i = (\alpha_{i-1}, \alpha_i) \subset [0,1]$ with $\alpha_0 = 0$, $\alpha_n = 1$ and $\Omega_j \cap \Omega_k = \varnothing$, $j \neq k$, i.e.

$$\chi_i(\alpha) = \begin{cases} 1, & \alpha \in \Omega_i, \\ 0, & \alpha \notin \Omega_i. \end{cases} \tag{4.89}$$

Then, for $h_i \in \mathbb{R}$, let $PC_{\mathbb{R}}(0,1)$ denote the space of mean-zero, simple functions:

$$\left\{ g(\alpha) \in C^0(0,1) \text{ a.e.} \,\middle|\, g(\alpha) = \sum_{i=1}^{n} h_i\chi_i(\alpha) \text{ and } \sum_{i=1}^{n} h_i\mu(\Omega_i) = 0 \right\} \tag{4.90}$$

where $\mu(\Omega_i) = \alpha_i - \alpha_{i-1}$, the Lebesgue measure of Ω_i. Observe that for $u_0'(\alpha) \in PC_{\mathbb{R}}(0,1)$ and $\lambda \neq 0$, (3.14), (4.89) and (4.90) imply that

$$\bar{\mathcal{K}}_i(t) = \sum_{j=1}^{n} (1 - \lambda\eta(t)h_j)^{-i-\frac{1}{\lambda}} \mu(\Omega_j). \tag{4.91}$$

We prove the following Theorem:

Theorem 4.92. *Consider the initial boundary value problem (1.1)-(1.2) for periodic $u_0'(\alpha) \in PC_{\mathbb{R}}(0,1)$. Let $T > 0$ and assume solutions are defined for all $t \in [0,T]$. Then, the representation formula (3.19) implies that no global $W^{1,\infty}(0,1)$ solution can exist if $T \geq t_*$, where $t_* = +\infty$ for $\lambda \in [0,+\infty)$ and $0 < t_* < +\infty$ otherwise. In addition, $\lim_{t\uparrow t_*} \|u_x(x,t)\|_1 = +\infty$ if $\lambda \in (-\infty,-1)$, while*

$$\lim_{t\uparrow t_*} \|u_x(x,t)\|_p = \begin{cases} C, & -\frac{1}{p} \leq \lambda < 0, \\ +\infty, & -1 \leq \lambda < -\frac{1}{p} \end{cases}$$

for $p \geq 1$, $\lambda \in [-1,0)$ and $C \in \mathbb{R}^+$ that depend on the choice of λ and p.

Proof. Let C denote a generic constant which may depend on λ and p. Since

$$u_0'(\alpha) = \sum_{i=1}^{n} h_i \chi_i(\alpha), \tag{4.93}$$

for $h_i \in \mathbb{R}$ as in (4.90), then (3.15) and (4.91) give

$$\gamma_\alpha(\alpha,t)^{-\lambda} = (1 - \lambda\eta(t) \sum_{i=1}^{n} h_i \chi_i(\alpha)) \left(\sum_{i=1}^{n} (1 - \lambda\eta(t)h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i) \right)^{\lambda} \tag{4.94}$$

for $\eta \in [0, \eta_*)$, η_* as defined in (4.1) and

$$\begin{cases} M_0 = \max_i h_i > 0, \\ m_0 = \min_i h_i < 0. \end{cases} \tag{4.95}$$

Let $\mathcal{I}_{max}$ and $\mathcal{I}_{min}$ denote the sets of indexes for the intervals $\overline{\Omega}_i$ and $\underline{\Omega}_i$ respectively, defined by $\overline{\Omega}_i \equiv \{\overline{\alpha} \in [0,1] \,|\, u_0'(\overline{\alpha}) = M_0\}$ and $\underline{\Omega}_i \equiv \{\underline{\alpha} \in [0,1] \,|\, u_0'(\underline{\alpha}) = m_0\}$.

Global estimates for $\lambda \in (0,+\infty)$.

Let $\lambda \in (0,+\infty)$ and $\eta_* = \frac{1}{\lambda M_0}$. Using the above definitions, we may write

$$1 - \lambda\eta(t) \sum_{i=1}^{n} h_i \chi_i(\alpha) = 1 - \lambda\eta(t) \left(\sum_{i \in \mathcal{I}_{max}} M_0 \chi_i(\alpha) + \sum_{i \notin \mathcal{I}_{max}} h_i \chi_i(\alpha) \right) \tag{4.96}$$

and

$$\begin{aligned} \sum_{i=1}^{n} (1 - \lambda\eta(t)h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i) = & \sum_{i \in \mathcal{I}_{max}} (1 - \lambda\eta(t)M_0)^{-\frac{1}{\lambda}} \mu(\overline{\Omega}_i) \\ & + \sum_{i \notin \mathcal{I}_{max}} (1 - \lambda\eta(t)h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i). \end{aligned} \tag{4.97}$$

Then, for fixed $i \in \mathcal{I}_{max}$ choosing $\overline{\alpha} \in \overline{\Omega}_i$ and substituting into (4.96), we find

$$1 - \lambda\eta(t) \sum_{i=1}^{n} h_i \chi_i(\overline{\alpha}) = 1 - \lambda\eta(t)M_0. \tag{4.98}$$

Using (4.97), (4.98) and (4.94), we see that, for $\eta \in [0, \eta_*)$,

$$\gamma_\alpha(\overline{\alpha}, t) = \left[\sum_{i \in \mathcal{I}_{max}} \mu(\overline{\Omega}_i) + (1 - \lambda\eta(t) M_0)^{\frac{1}{\lambda}} \sum_{i \notin \mathcal{I}_{max}} (1 - \lambda\eta(t) h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i) \right]^{-1}. \tag{4.99}$$

Since $1 - \lambda\eta(t) u_0'(\alpha) > 0$ for all $\eta \in [0, \eta_*)$ and $\alpha \in [0, 1]$, (4.99) implies

$$\lim_{t \uparrow t_*} \gamma_\alpha(\overline{\alpha}, t) = \left(\sum_{i \in \mathcal{I}_{max}} \mu(\overline{\Omega}_i) \right)^{-1} > 0 \tag{4.100}$$

for some $t_* > 0$. However, (3.15), (3.17) and (4.93) give

$$dt = \left(1 - \lambda\eta(t) \sum_{i=1}^{n} h_i \chi_i(\alpha) \right)^{-2} \gamma_\alpha(\alpha, t)^{-2\lambda} d\eta \tag{4.101}$$

and so, for $\eta_* - \eta > 0$ small, (4.94), (4.97) and the above observation on $1 - \lambda\eta(t) u_0'(\alpha)$ yield, after integration, $t_* - t \sim C \int_\eta^{\eta_*} (1 - \lambda M_0 \sigma)^{-2} d\sigma$. Consequently, $t_* = +\infty$. Finally, (2.5)i), (4.50) and (4.100) yield

$$\lim_{t \to +\infty} \int_0^t M(s)\, ds = -\ln \left(\sum_{i \in \mathcal{I}_{max}} \mu(\overline{\Omega}_i) \right) > 0.$$

If $\alpha = \widetilde{\alpha} \in \Omega_i$ for some index $i \notin \mathcal{I}_{max}$, so that $1 - \lambda\eta(t) u_0'(\widetilde{\alpha}) = 1 - \lambda\eta(t)\tilde{h}$ for $\tilde{h} < M_0$, then (4.94) implies $\gamma_\alpha(\widetilde{\alpha}, t) \sim C(1 - \lambda\eta(t) M_0)^{\frac{1}{\lambda}} \to 0$ as $t \to +\infty$. Thus, by (4.50), we obtain

$$\lim_{t \to +\infty} \int_0^t u_x(\gamma(\widetilde{\alpha}, s), s)\, ds = -\infty.$$

We refer to appendix A for the case $\lambda = 0$.

L^p regularity for $p \in [1, +\infty]$ and $\lambda \in (-\infty, 0)$.

Suppose $\lambda \in (-\infty, 0)$ so that $\eta_* = \frac{1}{\lambda m_0}$. We now write

$$1 - \lambda\eta(t) \sum_{i=1}^{n} h_i \chi_i(\alpha) = 1 - \lambda\eta(t) \left(\sum_{i \in \mathcal{I}_{min}} m_0 \chi_i(\alpha) + \sum_{i \notin \mathcal{I}_{min}} h_i \chi_i(\alpha) \right) \tag{4.102}$$

and

$$\begin{aligned} \sum_{i=1}^{n} (1 - \lambda\eta(t) h_i)^{\frac{1}{|\lambda|}} \mu(\Omega_i) = & \sum_{i \in \mathcal{I}_{min}} (1 - \lambda\eta(t) m_0)^{\frac{1}{|\lambda|}} \mu(\underline{\Omega}_i) \\ & + \sum_{i \notin \mathcal{I}_{min}} (1 - \lambda\eta(t) h_i)^{\frac{1}{|\lambda|}} \mu(\Omega_i). \end{aligned} \tag{4.103}$$

Choose $\underline{\alpha} \in \underline{\Omega}_i$ for some $i \in \mathcal{I}_{min}$ and substitute into (4.102) to obtain

$$1 - \lambda\eta(t) \sum_{i=1}^{n} h_i \chi_i(\underline{\alpha}) = 1 - \lambda\eta(t) m_0. \tag{4.104}$$

Using (4.103) and (4.104) with (4.94) gives

$$\gamma_\alpha(\underline{\alpha}, t) = \left[\sum_{i \in \mathcal{I}_{min}} \mu(\underline{\Omega}_i) + \frac{\sum_{i \notin \mathcal{I}_{min}} (1 - \lambda\eta(t) h_i)^{\frac{1}{|\lambda|}} \mu(\Omega_i)}{(1 - \lambda\eta(t) m_0)^{\frac{1}{|\lambda|}}} \right]^{-1} \tag{4.105}$$

for $\eta \in [0, \eta_*)$. Since $1 - \lambda\eta(t)u_0'(\alpha) > 0$ for $\eta \in [0, \eta_*)$, $\alpha \in [0,1]$ and $\lambda < 0$, we have that $\lim_{t\uparrow t_*} \gamma_\alpha(\underline{\alpha}, t) = 0$ or, equivalently,

$$\lim_{t\to t_*} \int_0^t m(s)ds = -\infty$$

by (2.5)ii) and (4.50). The blow-up time $t_* > 0$ is now finite. Indeed, (4.94), (4.101) and (4.103) yield the estimate

$$dt \sim \left(\sum_{i\notin \mathcal{I}_{min}} (1 - \lambda\eta(t)h_i)^{\frac{1}{|\lambda|}} \mu(\Omega_i) \right)^{2\lambda} d\eta$$

for $\eta_* - \eta > 0$ small and $\lambda < 0$. Since $h_i > m_0$ for any $i \notin \mathcal{I}_{min}$, integration of the above implies a finite $t_* > 0$. Now, if $\alpha = \alpha' \in \Omega_i$ for some $i \notin \mathcal{I}_{min}$, then $u_0'(\alpha') = h'$ for $h' > m_0$. Following the argument in the $\lambda > 0$ case yields

$$\gamma_\alpha(\alpha', t) = \left(\sum_{i\notin \mathcal{I}_{min}} (1 - \lambda\eta(t)h_i)^{\frac{1}{|\lambda|}} \mu(\Omega_i) \right)^{-1} (1 - \lambda\eta(t)h')^{\frac{1}{|\lambda|}},$$

consequently $\lim_{t\uparrow t_*} \gamma_\alpha(\alpha', t) = C \in \mathbb{R}^+$ and so, by (4.50), $\int_0^t u_x(\gamma, s)\, ds$ remains finite as $t \uparrow t_*$ for every $\alpha' \neq \underline{\alpha}$ and $\lambda \in (-\infty, 0)$.
Lastly, we look at L^p regularity of u_x for $p \in [1, +\infty)$ and $\lambda \in (-\infty, 0)$. From (3.15) and (3.19),

$$|u_x(\gamma(\alpha,t),t)|^p\, \gamma_\alpha(\alpha,t) = \frac{\mathcal{K}_0(\alpha,t)|\mathcal{J}(\alpha,t)^{-1} - \bar{\mathcal{K}}_0(t)^{-1}\bar{\mathcal{K}}_1(t)|^p}{|\lambda\eta(t)|^p\, \bar{\mathcal{K}}_0(t)^{2\lambda p+1}}$$

for $t \in [0, t_*)$ and $p \in \mathbb{R}$. Then, integrating in α and using (4.91) gives

$$\|u_x(x,t)\|_p^p = \frac{1}{|\lambda\eta(t)|^p} \left(\sum_{i=1}^n (1 - \lambda\eta(t)h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i) \right)^{-(2\lambda p+1)}$$
$$\sum_{j=1}^n \left\{ (1 - \lambda\eta(t)h_j)^{-\frac{1}{\lambda}} \left| (1 - \lambda\eta(t)h_j)^{-1} - \frac{\sum_{i=1}^n (1 - \lambda\eta(t)h_i)^{-1-\frac{1}{\lambda}} \mu(\Omega_i)}{\sum_{i=1}^n (1 - \lambda\eta(t)h_i)^{-\frac{1}{\lambda}} \mu(\Omega_i)} \right|^p \mu(\Omega_j) \right\}$$

for $p \in [1, +\infty)$. Splitting each sum above into the indexes $i, j \in \mathcal{I}_{min}$ and $i, j \notin \mathcal{I}_{min}$, we obtain, for $\eta_* - \eta > 0$ small,

$$\|u_x(x,t)\|_p^p \sim C\mathcal{J}(\underline{\alpha},t)^{-\frac{1}{\lambda}} \left| \mathcal{J}(\underline{\alpha},t)^{-1} - C\left(\mathcal{J}(\underline{\alpha},t)^{-1-\frac{1}{\lambda}} + C \right) \right|^p$$
$$+ C \sum_{j\notin \mathcal{I}_{min}} \left\{ (1 - \lambda\eta h_j)^{-\frac{1}{\lambda}} \left| (1 - \lambda\eta h_j)^{-1} - C\left(\mathcal{J}(\underline{\alpha},t)^{-1-\frac{1}{\lambda}} + C \right) \right|^p \mu(\Omega_j) \right\}$$

where $\lambda \in (-\infty, 0)$, $\mathcal{J}(\underline{\alpha}, t) = 1 - \lambda\eta(t)m_0$ and the constant $C > 0$ may now also depend on $p \in [1, +\infty)$. Suppose $\lambda \in [-1, 0)$, then $-1 - \frac{1}{\lambda} \geq 0$ and the above implies

$$\|u_x(x,t)\|_p^p \sim C\mathcal{J}(\underline{\alpha},t)^{-\left(p+\frac{1}{\lambda}\right)} + g(t) \tag{4.106}$$

for $g(t)$ a bounded function on $[0, t_*)$ with finite, non-negative limit as $t \uparrow t_*$. On the other hand, if $\lambda \in (-\infty, -1)$ then $-1 - \frac{1}{\lambda} < 0$ and

$$\|u_x(x,t)\|_p^p \sim C\mathcal{J}(\underline{\alpha},t)^{-\left(p+\frac{1}{\lambda}\right)} \tag{4.107}$$

holds instead. The last part of the Theorem follows from (4.106) and (4.107) as $t \uparrow t_*$. See §5.2 for examples. □

4.2.2. Piecewise constant $u_0''(x)$.
When $u_0''(\alpha) \in PC_{\mathbb{R}}(0,1)$, the behaviour of solutions, in particular for $\lambda \in (1/2, 1]$, can be rather different than the one described in theorems 4.7 and 4.92. Below, we consider a particular choice of data u_0 with a finite jump discontinuity in u_0'' at the point $\overline{\alpha}$ where u_0' attains its maximum M_0. We find that the solution undergoes a two-sided blow-up in finite-time for $\lambda \in (1/2, +\infty)$. In particular, this signifies the formation of singularities in stagnation point-form solutions to the 2D incompressible Euler equations ($\lambda = 1$) ([4], [22], [20]). For $\lambda \in (-\infty, 0)$, we find that a one-sided blow-up occurs at a finite number of locations in the domain.
Let

$$u_0(\alpha) = \begin{cases} 2\alpha^2 - \alpha, & \alpha \in [0, 1/2], \\ -2\alpha^2 + 3\alpha - 1, & \alpha \in (1/2, 1] \end{cases} \tag{4.108}$$

so that $M_0 = 1$ and $m_0 = -1$ occur at $\overline{\alpha} = 1/2$ and $\underline{\alpha} = \{0, 1\}$ respectively. Then $\mathcal{J}(\overline{\alpha}, t) = 1 - \lambda\eta(t)$, $\mathcal{J}(\underline{\alpha}, t) = 1 + \lambda\eta(t)$ and $\eta_* = \frac{1}{|\lambda|}$ for $\lambda \neq 0$. Using (4.108), we find

$$\bar{\mathcal{K}}_0(t) = \begin{cases} \frac{\mathcal{J}(\overline{\alpha},t)^{1-\frac{1}{\lambda}} - \mathcal{J}(\underline{\alpha},t)^{1-\frac{1}{\lambda}}}{2(1-\lambda)\eta(t)}, & \lambda \in \mathbb{R}\backslash\{0,1\}, \\ \frac{1}{2\eta(t)} \ln\left(\frac{\eta_* + \eta(t)}{\eta_* - \eta(t)}\right), & \lambda = 1 \end{cases} \tag{4.109}$$

and

$$\bar{\mathcal{K}}_1(t) = \frac{\mathcal{J}(\overline{\alpha}, t)^{-\frac{1}{\lambda}} - \mathcal{J}(\underline{\alpha}, t)^{-\frac{1}{\lambda}}}{2\eta(t)}, \qquad \lambda \neq 0. \tag{4.110}$$

If $\lambda \in (-\infty, 0)$, a one-sided blow-up, $m(t) \to -\infty$, follows trivially from (3.19), (4.109)i) and (4.110) as t approaches a finite $t_* > 0$ whose existence is guaranteed, in the limit as $\eta \uparrow \eta_*$, by (3.17) and (4.109)i). On the other hand, if $\lambda \in (0, +\infty)$ and $\eta_* - \eta > 0$ is small,

$$\bar{\mathcal{K}}_0(t) \sim \begin{cases} \frac{\lambda}{2(1-\lambda)} \mathcal{J}(\overline{\alpha}, t)^{1-\frac{1}{\lambda}}, & \lambda \in (0, 1), \\ \frac{\lambda}{2^{\frac{1}{\lambda}}(\lambda - 1)}, & \lambda \in (1, +\infty) \\ -C \log(\eta_* - \eta(t)), & \lambda = 1 \end{cases} \tag{4.111}$$

and

$$\bar{\mathcal{K}}_1(t) \sim \frac{\lambda}{2\mathcal{J}(\overline{\alpha}, t)^{\frac{1}{\lambda}}}. \tag{4.112}$$

For $\alpha = \overline{\alpha}$, the above estimates and (3.19) imply that, as $\eta \uparrow \eta_*$,

$$M(t) = u_x(\gamma(\overline{\alpha}, t), t) \to \begin{cases} 0, & \lambda \in (0, 1/2), \\ +\infty, & \lambda \in (1/2, +\infty). \end{cases}$$

Furthermore, for $\alpha \neq \overline{\alpha}$,

$$u_x(\gamma(\alpha, t), t) \to \begin{cases} 0, & \lambda \in (0, 1/2), \\ -\infty, & \lambda \in (1/2, +\infty). \end{cases}$$

For the threshold parameter $\lambda = 1/2$, $u_x(\gamma, t) \to -1$ as $\eta \uparrow 2$ for $\alpha \notin \{\overline{\alpha}, \underline{\alpha}\}$, whereas, $M(t) = u_x(\gamma(\overline{\alpha}, t), t) \equiv 1$ and $m(t) = u_x(\gamma(\underline{\alpha}, t), t) \equiv -1$. Finally, from (3.17) and (4.111)

$$t_* - t \sim \begin{cases} C \int_\eta^{\eta_*} (1 - \lambda\mu)^{2(\lambda-1)} d\mu, & \lambda \in (0, 1), \\ C(\eta_* - \eta)(2 - 2\log(\eta_* - \eta) + \ln^2(\eta_* - \eta)), & \lambda = 1, \\ C(\eta_* - \eta), & \lambda \in (1, +\infty), \end{cases}$$

and so $t_* = +\infty$ for $\lambda \in (0, 1/2]$ but $0 < t_* < +\infty$ when $\lambda \in (1/2, +\infty)$.

Remark 4.113. By following an argument analogous to that of §4.2.1 for piecewise constant u_0', it can be shown that the results from the above example extend to arbitrary data with piecewise constant u_0''. In fact, if instead of having u_0' piecewise linear in $[0, 1]$, the above results extend to continuous u_0' behaving linearly only in a small neighbourhood of $\overline{\alpha}_i$ for $\lambda > 0$. Similarly for parameters $\lambda < 0$ and u_0' locally linear near $\underline{\alpha}_j$. Further details on the generalization of this result, as well as new results for data with arbitrary curvature near the locations in question, will be presented in a forthcoming paper.

Remark 4.114. We recall that if $\lambda \in [1/2, 1)$ and $u_0'''(x) \in L_{\mathbb{R}}^{\frac{1}{2(1-\lambda)}}(0, 1)$, then u persists globally in time ([18]). This result does not contradict the above blow-up example. Indeed, if $u_0''' \in L_{\mathbb{R}}^{\frac{1}{2(1-\lambda)}}$ for $\lambda \in [1/2, 1)$, then u_0'' is an absolutely continuous function on $[0, 1]$, and hence continuous. However, in the case just considered, u_0'' is, of course, not continuous.

Remark 4.115. From Theorem 4.7, which examines a family of smooth $u_0 \in C_{\mathbb{R}}^{\infty}$, notice that $\lambda_* = 1$ acts as the threshold parameter between solutions that vanish at $t = +\infty$ for $\lambda \in (0, \lambda_*)$ and those which blow-up in finite-time when $\lambda \in (\lambda_*, +\infty)$, while for $\lambda_* = 1$, u_x converges to a non-trivial steady-state as $t \to +\infty$. In the example above with $u_0'' \in PC_{\mathbb{R}}$, we have the corresponding behavior at $\lambda_* = 1/2$ instead. Interestingly enough, when $\lambda = 1/2$ or $\lambda = 1$, equation (1.1) i), iii) models stagnation point-form solutions to the 3D or 2D incompressible Euler equations respectively. An interesting question would be to examine the effect on blow-up of cusps in the graph of u_0', for $\lambda = 1/2$.

5. Examples

Examples 1-4 in §5.1 have $\lambda \in \{3, -5/2, 1, -1/2\}$, respectively, and are instances of theorems 4.7, 4.36 and 4.52. In these cases, we will use formula (3.20) and the MATHEMATICA software to aid in the closed-form evaluation of some of the integrals and the generation of plots. Furthermore, examples 5 and 6 in §5.2 are representatives of Theorem 4.92 for $\lambda = 1$ and -2. For simplicity, details of the computations in most examples are omitted. Finally, because solving the IVP (3.17) is generally a difficult task, the plots in this section (with the exception of figure $2A$) will depict $u_x(\gamma(\alpha, t), t)$ for fixed $\alpha \in [0, 1]$ against the variable $\eta(t)$, rather than t. Figure $2A$ will however illustrate $u(x, t)$ versus $x \in [0, 1]$ for fixed $t \in [0, t_*)$.

5.1. Examples for theorems 4.7, 4.52 and 4.36

For examples 1-3, let $u_0(\alpha) = -\frac{1}{4\pi}\cos(4\pi\alpha)$. Then $u_0'(\alpha) = \sin(4\pi\alpha)$ attains its maximum $M_0 = 1$ at $\overline{\alpha}_i = \{1/8, 5/8\}$, while $m_0 = -1$ occurs at $\underline{\alpha}_j = \{3/8, 7/8\}$.

Example 1. Two-sided Blow-up for $\lambda = 3$. Let $\lambda = 3$, then $\eta_* = \frac{1}{\lambda M_0} = 1/3$ and for the time-dependent integrals we find that

$$\bar{\mathcal{K}}_0(t) = {}_2F_1\left[\frac{1}{6}, \frac{2}{3}; 1; 9\,\eta(t)^2\right] \to \frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)\,\Gamma\left(\frac{5}{6}\right)} \sim 1.84 \tag{5.1}$$

and

$$\int_0^1 \frac{u_0'\,d\alpha}{(1 - 3\eta(t)u_0')^{\frac{4}{3}}} = 2\eta(t)\,{}_2F_1\left[\frac{7}{6}, \frac{5}{3}; 2; 9\,\eta(t)^2\right] \to +\infty \tag{5.2}$$

as $\eta \uparrow 1/3$. Using (5.1) and (5.2) on (3.20), and taking the limit as $\eta \uparrow 1/3$, we see that $M(t) = u_x(\gamma(\overline{\alpha}_i, t), t) \to +\infty$ whereas, for $\alpha \neq \overline{\alpha}_i$, $u_x(\gamma(\alpha, t), t) \to -\infty$. The blow-up time $t_* \sim 0.54$ is obtained from (3.17) and (5.1). See figure 1*A*.

Example 2. Two-sided Blow-up for $\lambda = -5/2$. For $\lambda = -5/2$, $\eta_* = \frac{1}{\lambda m_0} = 2/5$. Then, we now have that

$$\bar{\mathcal{K}}_0(t) = {}_2F_1\left[-\frac{1}{5}, \frac{3}{10}; 1; \frac{25}{4}\eta(t)^2\right] \to \frac{\Gamma\left(\frac{9}{10}\right)}{\Gamma\left(\frac{7}{10}\right)\Gamma\left(\frac{6}{5}\right)} \sim 0.9 \tag{5.3}$$

and

$$\int_0^1 \frac{u_0' \, d\alpha}{(1 + 5\eta(t)u_0'/2)^{\frac{3}{5}}} = -\frac{3}{4}\eta(t)\, {}_2F_1\left[\frac{4}{5}, \frac{13}{10}; 2; \frac{25}{4}\eta(t)^2\right] \to -\infty \tag{5.4}$$

as $\eta \uparrow 2/5$. Plugging the above formulas into (3.20) and letting $\eta \uparrow 2/5$, we find that $m(t) = u_x(\gamma(\underline{\alpha}_j, t), t) \to -\infty$ while, for $\alpha \neq \underline{\alpha}_j$, $u_x(\gamma(\alpha, t), t) \to +\infty$. The blow-up time $t_* \sim 0.46$ is obtained from (3.17) and (5.3). See figure 1*B*.

Example 3. Global Existence for $\lambda = 1$. Let $\lambda = 1$, then

$$\bar{\mathcal{K}}_0(t) = \frac{1}{\sqrt{1 - \eta(t)^2}} \quad \text{and} \quad \int_0^1 \frac{u_0' \, d\alpha}{(1 - \eta(t)u_0')^2} = \frac{\eta(t)}{(1 - \eta(t)^2)^{\frac{3}{2}}} \tag{5.5}$$

both diverge to $+\infty$ as $\eta \uparrow \eta_* = 1$. Also, (5.5)i) and (3.17) imply $\eta(t) = \tanh t$, which we use on (3.20), along with (5.5), to obtain

$$u_x(\gamma(\alpha, t), t) = \frac{\tanh t - \sin(4\pi\alpha)}{\tanh t \sin(4\pi\alpha) - 1}.$$

Then, $M(t) = u_x(\gamma(\overline{\alpha}_i, t), t) \equiv 1$ and $m(t) = u_x(\gamma(\underline{\alpha}_j, t), t) \equiv -1$ while, for $\alpha \notin \{\overline{\alpha}_i, \underline{\alpha}_j\}$, $u_x(\gamma(\alpha, t), t) \to -1$ as $\eta \uparrow 1$. Finally, $\eta(t) = \tanh t$ yields $t_* = \lim_{\eta \uparrow 1} \operatorname{arctanh} \eta = +\infty$. It is also easy to see from the formulas in §3 and (5.5)i) that $I(t) \equiv -1$ for $I(t)$ the nonlocal term (1.1)iii). See figure 1*C*.

Example 4. One-sided Blow-up for $\lambda = -1/2$. For $\lambda = -1/2$ (HS equation), let $u_0 = \cos(2\pi\alpha) + 2\cos(4\pi\alpha)$. Then, the least value $m_0 < 0$ of u_0' and the location $\underline{\alpha} \in [0, 1]$ where it occurs are given, approximately, by $m_0 \sim -30$ and $\underline{\alpha} \sim 0.13$, respectively. Also, $\eta_* = -\frac{2}{m_0} \sim 0.067$ and

$$\bar{\mathcal{K}}_0(t) = 1 + \frac{17\pi^2\eta(t)^2}{2}, \qquad \int_0^1 u_0'(\alpha)\left(1 + \eta(t)\frac{u_0'(\alpha)}{2}\right) d\alpha = 17\pi^2\eta(t). \tag{5.6}$$

Then, (3.17) and (5.6)i) give $\eta(t) = \sqrt{\frac{2}{17\pi^2}} \tan\left(\pi\sqrt{\frac{17}{2}}\, t\right)$. Using these results on (3.20) yields, after simplification,

$$u_x(\gamma(\alpha, t), t) = \frac{\pi\left(2\sin(2\pi\alpha) + 8\sin(4\pi\alpha) + \sqrt{34}\tan\left(\pi\sqrt{\frac{17}{2}}t\right)\right)}{\sqrt{\frac{2}{17}}\tan\left(\pi\sqrt{\frac{17}{2}}\, t\right)(\sin(2\pi\alpha) + 4\sin(4\pi\alpha)) - 1}$$

for $0 \leq \eta < \eta_*$. We find that $m(t) = u_x(\gamma(\underline{\alpha}, t), t) \to -\infty$ as $\eta \uparrow \eta_*$, whereas, for $\alpha \neq \underline{\alpha}$, $u_x(\gamma(\alpha, t), t)$ remains finite. Finally, $t_* = t\,(-2/m_0) \sim 0.06$. See figure 1*D*.

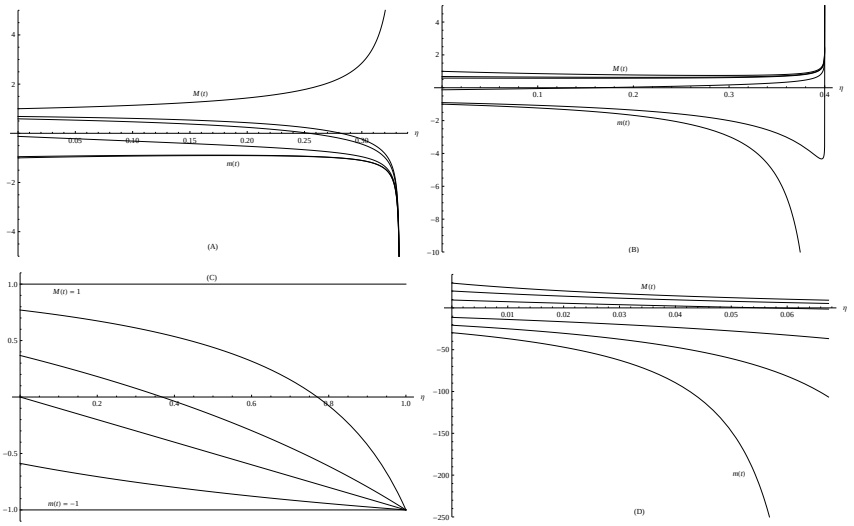

FIGURE 1. Figures A and B depict two-sided, everywhere blow-up of (3.20) for $\lambda = 3$ and $-5/2$ (Examples 1 and 2) as $\eta \uparrow 1/3$ and $2/5$, respectively. Figure C (Example 3) represents global existence in time for $\lambda = 1$, whereas figure D (Example 4) illustrates one-sided, discrete blow-up for $\lambda = -1/2$ as $\eta \uparrow 0.067$.

5.2. Examples for Theorem 4.92

For examples 5 and 6 below, we let

$$u_0(\alpha) = \begin{cases} -\alpha, & 0 \le \alpha < 1/4, \\ \alpha - 1/2, & 1/4 \le \alpha < 3/4. \\ 1 - \alpha, & 3/4 \le \alpha \le 1. \end{cases} \tag{5.7}$$

Then, $M_0 = 1$ occurs when $\alpha \in [1/4, 3/4)$, $m_0 = -1$ for $\alpha \in [0, 1/4) \cup [3/4, 1]$ and $\eta_* = \frac{1}{|\lambda|}$ for $\lambda \neq 0$. Also, notice that (5.7) is odd about the midpoint $\alpha = 1/2$ and it vanishes at the end-points (as it should due to periodicity). As a result, uniqueness of solution to (2.2) implies that $\gamma(0,t) \equiv 0$ and $\gamma(1,t) \equiv 1$ for as long as u is defined.

Example 5. Global estimates for $\lambda = 1$. Using (5.7), we find that $\bar{\mathcal{K}}_0(t) = (1-\eta(t)^2)^{-1}$ for $0 \le \eta < \eta_* = 1$. Then (3.15) implies $\gamma_\alpha(\alpha,t) = \frac{1-\eta(t)^2}{1-\eta(t)u_0'(\alpha)}$, or

$$\gamma(\alpha,t) = \begin{cases} (1-\eta(t))\alpha, & 0 \le \alpha < 1/4, \\ \alpha + \eta(t)(\alpha - 1/2), & 1/4 \le \alpha < 3/4, \\ \alpha + \eta(t)(1-\alpha), & 3/4 \le \alpha \le 1 \end{cases} \tag{5.8}$$

after integrating and using (5.7) and $\gamma(0,t) \equiv 0$. Since $\dot{\gamma} = u \circ \gamma$, we have that

$$u(\gamma(\alpha,t),t) = \begin{cases} -\alpha\dot{\eta}(t), & 0 \le \alpha < 1/4, \\ (\alpha - 1/2)\dot{\eta}(t), & 1/4 \le \alpha < 3/4 \\ (1-\alpha)\dot{\eta}(t), & 3/4 \le \alpha \le 1 \end{cases} \tag{5.9}$$

where, by (3.17) and $\bar{\mathcal{K}}_0$ above, $\dot{\eta}(t) = (1-\eta(t)^2)^2$. Now, (5.8) lets us solve for $\alpha = \alpha(x,t)$, the inverse Lagrangian map, as

$$\alpha(x,t) = \begin{cases} \frac{x}{1-\eta(t)}, & 0 \le x < \frac{1-\eta(t)}{4}, \\ \frac{2x+\eta(t)}{2(1+\eta(t))}, & \frac{1-\eta(t)}{4} \le x < \frac{3+\eta(t)}{4}, \\ \frac{x-\eta(t)}{1-\eta(t)}, & \frac{3+\eta(t)}{4} \le x \le 1, \end{cases} \tag{5.10}$$

which we use on (5.9) to obtain the corresponding Eulerian representation

$$u(x,t) = \begin{cases} -(1-\eta(t))(1+\eta(t))^2 x, & 0 \le x < \frac{1-\eta(t)}{4}, \\ \frac{1}{2}(1+\eta(t))(1-\eta(t))^2(2x-1), & \frac{1-\eta(t)}{4} \le x < \frac{3+\eta(t)}{4}, \\ (1-\eta(t))(1+\eta(t))^2(1-x), & \frac{3+\eta(t)}{4} \le x \le 1, \end{cases} \tag{5.11}$$

which in turn yields

$$u_x(x,t) = \begin{cases} -(1-\eta(t))(1+\eta(t))^2, & 0 \le x < \frac{1-\eta(t)}{4}, \\ (1+\eta(t))(1-\eta(t))^2, & \frac{1-\eta(t)}{4} \le x < \frac{3+\eta(t)}{4}, \\ -(1-\eta(t))(1+\eta(t))^2, & \frac{3+\eta(t)}{4} \le x \le 1. \end{cases} \tag{5.12}$$

Finally, solving the IVP for η we obtain $t(\eta) = \frac{1}{2}\left(\operatorname{arctanh}(\eta) + \frac{\eta}{1-\eta^2}\right)$, so that (3.18) gives $t_* = \lim_{\eta\uparrow 1} t(\eta) = +\infty$. See figure 2*A* below.

Remark 5.13. The vanishing of the characteristics in example 5 greatly facilitates the computation of an explicit solution formula for $\dot{\gamma}(\alpha,t) = u(\gamma(\alpha,t),t)$. However, as it is generally the case, $\gamma(0,t)$ may not be identically zero. In that case, integration of (3.15) now yields

$$\gamma(\alpha,t) = \gamma(0,t) + \frac{1}{\bar{\mathcal{K}}_0(t)} \int_0^\alpha \frac{dy}{\mathcal{J}(y,t)^{\frac{1}{\lambda}}}, \tag{5.14}$$

which we differentiate in time to obtain

$$u(\gamma(\alpha,t),t) = \dot{\gamma}(0,t) + \frac{d}{dt}\left(\frac{1}{\bar{\mathcal{K}}_0(t)} \int_0^\alpha \mathcal{K}_0(y,t)dy\right). \tag{5.15}$$

In order to determine the time-dependent function $\dot{\gamma}(0,t)$, we may use, for instance, the 'conservation in mean' condition[13]

$$\int_0^1 u_0(x)dx = \int_0^1 u(x,t)dx = \int_0^1 u(\gamma(\alpha,t),t)\gamma_\alpha(\alpha,t)d\alpha. \tag{5.16}$$

Let us assume (5.16) holds. Then, multiplying (5.15) by the mean-one function γ_α in (3.15), integrating in α and using (5.16), yields

$$\dot{\gamma}(0,t) = \int_0^1 u_0(\alpha)d\alpha - \int_0^1 \frac{\mathcal{K}_0(\alpha,t)}{\bar{\mathcal{K}}_0(t)} \frac{d}{dt}\left(\frac{1}{\bar{\mathcal{K}}_0(t)} \int_0^\alpha \mathcal{K}_0(y,t)dy\right) d\alpha. \tag{5.17}$$

[13]In [20], the authors showed that (5.16) follows naturally from the study of spatially periodic, stagnation point-form solutions to the n dimensional Euler equations with spatially periodic pressure term.

Omitting the details of the computations, (5.17) may, in turn, be written as

$$\dot{\gamma}(0,t) = \int_0^1 u_0(\alpha)d\alpha + \frac{\bar{\mathcal{K}}_0(t)^{-2(1+\lambda)}}{\lambda\eta(t)} \left(\frac{\bar{\mathcal{K}}_0(t)\bar{\mathcal{K}}_1(t)}{2} - \int_0^1 \mathcal{K}_0(\alpha,t) \int_0^\alpha \mathcal{K}_1(y,t)\, dy\, d\alpha \right) \tag{5.18}$$

The above and (5.15) yield a representation formula for $u(\gamma,t)$. Integrating (5.18) in time and using (5.14) gives an expression for the characteristics γ. Finally, we remark that under Dirichlet boundary conditions and/or using initial data u_0 which is odd about the midpoint ([7], [23]), a general formula for $u(\gamma,t)$ can be obtained from (5.15) by simply setting $\dot{\gamma}(0,t) \equiv 0$.

Example 6. Finite-time blow-up for $\lambda = -2$. Using (5.7) and $\lambda = -2$,

$$\bar{\mathcal{K}}_0(t) = \frac{\sqrt{1-2\eta(t)} + \sqrt{1+2\eta(t)}}{2} \quad \text{and} \quad \int_0^1 \frac{u_0'(\alpha)\, d\alpha}{\mathcal{J}(\alpha,t)^{1+\frac{1}{\lambda}}} = \frac{d\bar{\mathcal{K}}_0(t)}{d\eta}$$

for $\eta \in [0,\eta_*)$ and $\eta_* = 1/2$. Then, (3.20) yields

$$u_x(\gamma(\alpha,t),t) = \begin{cases} M(t) = \frac{\left(\sqrt{1-2\eta(t)}+\sqrt{1+2\eta(t)}\right)^3}{8(1+2\eta(t))\sqrt{1-2\eta(t)}}, & \alpha \in [1/4, 3/4), \\ m(t) = -\frac{\left(\sqrt{1-2\eta(t)}+\sqrt{1+2\eta(t)}\right)^3}{8(1-2\eta(t))\sqrt{1+2\eta(t)}}, & \alpha \in [0,1/4) \cup [3/4,1], \end{cases} \tag{5.19}$$

so that $M(t) \to +\infty$ and $m(t) \to -\infty$ as $\eta \uparrow 1/2$. The finite blow-up time $t_* > 0$ is obtained from (3.17) and $\bar{\mathcal{K}}_0$ above. We find

$$t(\eta) = \frac{1}{6\eta^3}\left(\eta^2\left(6 - 4\sqrt{1-4\eta^2}\right) + \sqrt{1-4\eta^2} - 1\right),$$

so that $t_* = t(1/2) = 2/3$. See figure 2B below.

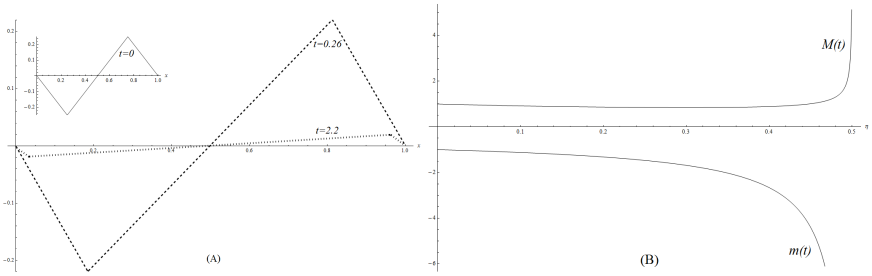

FIGURE 2. In figure A, (5.11) vanishes as $t \to +\infty$, while figure B depicts two-sided blow-up of (5.19) as $\eta \uparrow \eta_* = 1/2$.

Appendix A. Global existence for $\lambda = 0$ and smooth data.

Setting $\lambda = 0$ in (3.3) gives

$$(\ln \gamma_\alpha)^{\cdot\cdot} = -\int_0^1 u_x^2 dx = I(t).$$

Then, integrating twice in time and using $\dot{\gamma}_\alpha = (u_x(\gamma,t))\gamma_\alpha$ and $\gamma_\alpha(\alpha,0)=1$, yields

$$\gamma_\alpha(\alpha,t) = e^{tu_0'(\alpha)} e^{\int_0^t (t-s)I(s)ds}. \tag{A.1}$$

Since γ_α has mean one in $[0,1]$, integrate (A.1) in α to obtain

$$e^{\int_0^t (t-s)I(s)ds} = \left(\int_0^1 e^{tu_0'(\alpha)} d\alpha\right)^{-1}. \tag{A.2}$$

Combining this with (A.1) gives $\gamma_\alpha(\alpha,t) = e^{tu_0'(\alpha)} \left(\int_0^1 e^{tu_0'(\alpha)} d\alpha\right)^{-1}$, a bounded expression for γ_α which we differentiate w.r.t. to t, to get

$$u_x(\gamma(\alpha,t),t) = u_0'(\alpha) - \frac{\int_0^1 u_0'(\alpha) e^{tu_0'(\alpha)} d\alpha}{\int_0^1 e^{tu_0'(\alpha)} d\alpha} \tag{A.3}$$

and so

$$0 \le u_0'(\alpha) - u_x(\gamma(\alpha,t),t) \le \int_0^1 u_0'(\alpha)\, e^{tu_0'(\alpha)} d\alpha, \qquad t \ge 0.$$

The cases where $u_0', u_0'' \in PC_{\mathbb{R}}$ are analogous, and follow from the above.

Appendix B. Proof of lemma 4.5.

For the series (4.2), we have the following convergence results [9]:

- Absolute convergence for all $|z| < 1$.
- Absolute convergence for $|z| = 1$ if $Re(a+b-c) < 0$.
- Conditional convergence for $|z| = 1$, $z \neq 1$ if $0 \le Re(a+b-c) < 1$.
- Divergence if $|z| = 1$ and $1 \le Re(a+b-c)$.

Furthermore, consider the identities [9]:

$$\frac{d}{dz} {}_2F_1\left[a,b;c;z\right] = \frac{ab}{c}\, {}_2F_1\left[a+1,b+1;c+1;z\right], \quad {}_2F_1\left[a,b;b;z\right] = (1-z)^{-a}, \tag{B.1}$$

as well as the contiguous relations

$$z\, {}_2F_1\left[a+1,b+1;c+1;z\right] = \frac{c}{a-b}\left({}_2F_1\left[a,b+1;c;z\right] - {}_2F_1\left[a+1,b;c;z\right]\right) \tag{B.2}$$

and

$${}_2F_1\left[a,b;c;z\right] = \frac{b}{b-a}\, {}_2F_1\left[a,b+1;c;z\right] - \frac{a}{b-a}\, {}_2F_1\left[a+1,b;c;z\right] \tag{B.3}$$

for $b \neq a$. For simplicity, we prove the lemma for $\beta_0 = 0$ and write $F = {}_2F_1$.
Set $a = 1/2$, $c = 3/2$ and $z = -\frac{C_0\beta^2}{\epsilon}$. Then, the assumptions in the lemma imply that $-1 \le z \le 0$ and $a+b-c = b-1 < 1$, so that (B.1)i) and the chain rule give

$$\frac{d}{d\beta}\left\{\beta F\left[a,b;c;z\right]\right\} = \frac{2b}{3}\left(zF\left[a+1,b+1;c+1;z\right]\right) + F\left[a,b;c;z\right]. \tag{B.4}$$

But for $b \neq a = \frac{1}{2}$, (B.2) and (B.3) imply

$$zF\left[a+1,b+1;c+1;z\right] = \frac{3}{1-2b}\left(F\left[a,b+1;c;z\right] - F\left[b,c;c;z\right]\right)$$

and

$$F\left[a,b;c;z\right] = \frac{2b}{2b-1} F\left[a,b+1;c;z\right] - \frac{1}{2b-1} F\left[b,c;c;z\right].$$

Substituting the two above into (B.4) and using (B.1)ii)[14] yields our result. □

Appendix C. Proof of (2.5).

We prove (2.5) for $\lambda > 0$. The case of parameter values $\lambda < 0$ follows similarly. Suppose $\lambda > 0$ and set $\eta_\epsilon = \frac{1}{\lambda M_0 + \epsilon}$ for arbitrary $\epsilon > 0$. Then $0 < \eta_\epsilon < \eta_*$ for $\eta_* = \frac{1}{\lambda M_0}$. Also, due to the definition of M_0,

$$1 - \lambda\eta_\epsilon u_0'(\alpha) = \frac{\epsilon + \lambda(M_0 - u_0'(\alpha))}{\lambda M_0 + \epsilon} > 0$$

for all $\alpha \in [0,1]$, while $1 - \lambda\eta_\epsilon u_0'(\alpha) = 0$ only if $\epsilon = 0$ and $\alpha = \overline{\alpha}_i$. We conclude that

$$1 - \lambda\eta(t)u_0'(\alpha) > 0 \tag{C.1}$$

for all $0 \le \eta(t) < \eta_*$ and $\alpha \in [0,1]$. But $u_0'(\alpha) \le M_0$, or equivalently

$$u_0'(\alpha)(1 - \lambda\eta(t)M_0) \le M_0(1 - \lambda\eta(t)u_0'(\alpha)),$$

therefore (C.1) and $u_0'(\overline{\alpha}_i) = M_0$, $1 \le i \le m$, yield

$$\frac{u_0'(\alpha)}{\mathcal{J}(\alpha,t)} \le \frac{u_0'(\overline{\alpha}_i)}{\mathcal{J}(\overline{\alpha}_i,t)} \tag{C.2}$$

for $0 \le \eta < \eta_*$ and

$$\mathcal{J}(\alpha,t) = 1 - \lambda\eta(t)u_0'(\alpha), \qquad \mathcal{J}(\overline{\alpha}_i,t) = 1 - \lambda\eta(t)M_0.$$

The representation formula (3.20) and (C.2) then imply

$$u_x(\gamma(\overline{\alpha}_i,t),t) \ge u_x(\gamma(\alpha,t),t) \tag{C.3}$$

for $0 \le \eta(t) < \eta_*$ and $\alpha \in [0,1]$. Finally, one can easily see from (3.20) that, as long as the solution exists,

$$u_x(\gamma(\alpha_1,t),t) = u_x(\gamma(\alpha_2,t),t) \quad \Leftrightarrow \quad u_0'(\alpha_1) = u_0'(\alpha_2) \tag{C.4}$$

for all $\alpha_1, \alpha_2 \in [0,1]$. Then, (2.5)i) follows by using definition (2.3) and (in)equality (C.3). Likewise, $u_0'(\alpha) \ge m_0 = u_0'(\underline{\alpha}_j)$, (3.20) and (C.1) imply (2.5)ii). Similarly for $\lambda < 0$, (C.1) holds with $\eta_* = \frac{1}{\lambda m_0} > 0$ instead. Both (2.5)i), ii) then follow as above. □

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