NONLINEAR WAVE EQUATIONS ARISING IN MODELING OF SOME STRAIN-HARDENING STRUCTURES

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A class of nonlinear wave equations of p-Laplacian type are presented based on a generalized Hooke's law. These equations can be used to model vibration of rods, beams, and plates made of heat treated metals that satisfy the power-law stress-strain relationship in the framework of small strain theory. The examples include some rods and Euler-type beams in the form of a single stainless steel fibre of the hybrid stainless steel assembly used in transportation industry for lighter and more crashworthy vehicles. These metals are special cases of nonlinear strain-hardening elastic-plastic materials. Some finite element and finite difference schemes are also presented to fully discretize the wave equations and obtain numerical approximations, including linear and cubic finite elements, and some iterative finite difference schemes such as Newmark, Runge-Kutta,and others. The numerical results are analyzed and compared with some analytical solutions.

1 Generalized Hooke's law for Ludwick materials

1.1 One dimensional version

It is well known that, in uniaxial state, the stress and strain relation for the power-law plastic material, or sometimes referred to as the Ludwick material, is given by

$$\sigma = K|\epsilon|^{n-1}\epsilon, 0 < n \leq 1, \tag{1}$$

where σ is the stress, ϵ the strain, K and n are engineering constants with values depending on a specific material. This is within the small deformation theory. For a given metal or alloy, K and n depend on the heat treatment received by the metal or alloy. The parameter n is called the strain-hardening exponent for the material. When $n = 1$, equation (1) reduces to the Hooke's law for linear elastic material and the constant K equals the corresponding Young's modulus E. Table 1 is a list of some experimental values of K and n, see, e.g., Shackelford [12].

1.2 Higher dimensional versions

In the following, bold letters are used to denote for vectors or matrices. A vector is a single row matrix. The transpose of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}^\tau$, and the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{u}\mathbf{v}^\tau$. The time derivative $\frac{\partial \mathbf{u}}{\partial t}$ is denoted by $\dot{\mathbf{u}}$. Let the displacement vector be $\mathbf{u} = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$, let $\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_y = \frac{\partial v}{\partial y}$, $\epsilon_z = \frac{\partial w}{\partial z}$, $\gamma_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$, $\gamma_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$, $\gamma_{zx} = \frac{1}{2}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z})$ be the strain components, and let σ_x, σ_y,σ_z, τ_{xy}, τ_{yz},τ_{zx} denote the corresponding stress components. Let

$$D(\mathbf{u}) = \begin{pmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{pmatrix}, \; and \; |D(\mathbf{u})| = \sqrt{\epsilon_x^2 + \epsilon_y^2 + +\epsilon_z^2 + 2\gamma_{xy}^2 + 2\gamma_{yz}^2 + 2\gamma_{zx}^2}. \tag{2}$$

Table 1. Experimental Data of K and n for some heat treated alloys.

Alloy	K (MPa)	n
HSSA	368	0.11
Pure Aluminum	900	0.07
304 stainless steel	1275	0.45

A three dimensional Hooke's law for an isotropic Ludwick type material can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{K|D(\mathbf{u})|^{n-1}}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{pmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}, \tag{3}$$

where K and ν are material constants. When $n = 1$, K equals the Young's modulus of linear elasticity and ν the corresponding Poisson's ratio. For plane stress problems, (3) can be simplified to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{K}{1-\nu^2}|D(\mathbf{u})|^{n-1} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}. \tag{4}$$

with the assumption that $\sigma_z = \sigma_{yz} = \sigma_{zx} = 0$. Similar version for plane strain can be obtained by setting $\epsilon_z = \epsilon_{yz} = \epsilon_{zx} = 0$. There are similar versions of higher dimensional stress-strain relations for power-law hardening material in the literature. See, e.g., Gao [7], and Giannakopoulos [5] etc.

2 The Lagrangian functional for Ludwick plastics bodies

2.1 Lagrangian for Ludwick material

The potential energy for a Ludwick elastic-plastic body occupying a three dimension body V can by defined by

$$U = \frac{1}{n+1}\int_V \sigma\epsilon^\tau dV, \tag{5}$$

where $\epsilon = (\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$, and $\sigma = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$. The Lagrangian energy functional $I(\mathbf{u})$ equals the kinetic energy T minus the elastic-plastic potential energy U plus the work done by an external force W. It can be written as

$$I(\mathbf{u}) = \frac{1}{2}\int_V \rho\dot{\mathbf{u}}\dot{\mathbf{u}}^\tau dV - \frac{1}{n+1}\int_V \sigma\epsilon^\tau dV + \int_V \mathbf{f}\mathbf{u}^\tau dV + \int_{\partial V} \mathbf{t}\mathbf{u}^\tau dS, \tag{6}$$

where ρ is the density, $\dot{\mathbf{u}} = (\dot{u}, \dot{v}, \dot{w})$ the velocity, $\mathbf{f} = (f_x, f_y, f_z)$, the body force, and $\mathbf{t} = (t_x, t_y, t_z)$ the surface force. This general form of the Lagrangian functional can be simplified in the following special cases.

2.2 Lagrangian for the rod

For an axial rod with cross-sectional area A, density ρ, and length L, subject to an axial force f, we have $m = \rho A$, $\sigma = (\sigma_x, 0, 0, 0, 0, 0)$, $\mathbf{u} = (u(x,t), 0, 0)$, $\sigma_x = K|\epsilon_x|^{n-1}\epsilon_x$, $\mathbf{f} = (f, 0, 0)$, so the Lagrangian energy functional is given by

$$I(u) = \frac{1}{2}\int_0^L \rho A\dot{u}^2 dx + \frac{1}{n+1}\int_0^L KA|\frac{\partial u}{\partial x}|^{n+1}dx - \int_0^L Afudx. \tag{7}$$

2.3 Lagrangian for the plane uniaxial strain body

For an Ludwick material body which has an uniform thickness t, occupies an area Ω in the xy plane, with density ρ, and subject to an axial load $\mathbf{f} = (f(x,y,t), 0, 0)$, we assume that displacement $\mathbf{u} = (u(x,y,t), 0, 0)$, strains $\epsilon_x = \frac{\partial u}{\partial x}, \gamma_{xy} = \frac{1}{2}\frac{\partial u}{\partial y}$, stress $\sigma_x = K({\epsilon_x}^2 + 2{\gamma_{xy}}^2)^{\frac{n-1}{2}}\epsilon_x$, $\tau_x = ({\epsilon_x}^2 + 2{\gamma_{xy}}^2)^{\frac{n-1}{2}}\gamma_{xy}$. So the Lagrangian energy functional is given by

$$I(u) = \frac{1}{2}\int_\Omega \rho t\dot{u}^2 dxdy + \frac{1}{n+1}\int_\Omega tK[(\frac{\partial u}{\partial x})^2 + \frac{1}{2}(\frac{\partial u}{\partial y})^2]^{\frac{n+1}{2}}dxdy - \int_\Omega tfudxdy. \tag{8}$$

2.4 Lagrangian for the Euler beam

For the a Ludwick Euler beam, it is assumed that $u(x,y,t) = -y\frac{\partial v}{\partial x}$ and $v = v(x,t)$, $\mathbf{f} = (0, r(x,t), 0)$, therefore $\epsilon_x = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2}$, $\epsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = 0$, and $\epsilon_y = \epsilon_{xz} = \epsilon_{yz} = \epsilon_z = 0$. The potential energy is given by

$$U = \frac{1}{n+1}\int_V \sigma_x\epsilon_x dV = \frac{1}{n+1}\int_0^L KI_n|\frac{\partial^2 v}{\partial x^2}|^{n+1}dx, \tag{9}$$

where $I_n = \int_A |y|^{n+1}dA$ is the second moment of inertia of the cross-section for the Ludwick material. The Lagrangian functional has the following form

$$I(v) = \frac{1}{2}\int_0^L \rho A\dot{v}^2 dx + \frac{1}{n+1}\int_0^L KI_n|\frac{\partial^2 v}{\partial x^2}|^{n+1}dx - \int_0^L rvdx. \tag{10}$$

2.5 Lagrangian for the Euler plate

Thirdly, for a flat Euler plate of thickness h occupying an area of Ω in the xy plane, the displacement vector is of the form $\mathbf{u} = (-z\frac{\partial w}{\partial x}(x,y,t), -z\frac{\partial w}{\partial y}(x,y,t), w(x,y,t))$, the body force $\mathbf{f} = (0, 0, f(x,y,t))$ and $\epsilon_x = -z\frac{\partial^2 w}{\partial x^2}$, $\epsilon_y = -z\frac{\partial^2 w}{\partial y^2}$, $\gamma_{xy} = -z\frac{\partial^2 w}{\partial xy}$, $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$. The Lagrangian functional is

$$I(w) = \frac{1}{2}\int_\Omega \rho h\dot{w}^\tau w dA + \frac{1}{n+1}\int_V \sigma\epsilon^\tau dV - \int_\Omega hfwdA, \tag{11}$$

where, by using (4), the potential energy $\frac{1}{n+1}\int_V \sigma\epsilon^\tau dV$ equals

$$\frac{D_n}{n+1}\int_\Omega |D(w)|^{n-1}[(\frac{\partial^2 w}{\partial x^2})^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + (\frac{\partial^2 w}{\partial y^2})^2 + (1-\nu)(\frac{\partial^2 w}{\partial xy})^2]dA, \tag{12}$$

$$D_n = \frac{Kh^{n+2}}{(n+2)2^{n+1}(1-\nu^2)},\ \textit{and}\ |D(w)| = \sqrt{(\frac{\partial^2 w}{\partial x^2})^2 + 2(\frac{\partial^2 w}{\partial x\partial y})^2 + (\frac{\partial^2 w}{\partial y^2})^2}. \tag{13}$$

The quantity D_n reduces to the flexural rigidity of the Euler elastic thin plate when $n = 1$.

Figure 1. Numerical solution of the rod equation(15)

3 Hamilton principle and the wave equations

The Hamilton's principle which deal with time dependent situations. Specifically, Hamilton's pricinple requires that we seek a displacement $\mathbf{u}$ so that for any time interval $[t_1, t_2]$, $\mathbf{u}(t_1) = \mathbf{u}(t_2)$ and $\dot{\mathbf{u}}(t_1) = \dot{\mathbf{u}}(t_2)$, and for all displacment of the form $\mathbf{u} + \tau\mathbf{v}$, where $\mathbf{v}(t_1) = \mathbf{v}(t_2) = \mathbf{0}$, $\dot{\mathbf{v}}(t_1) = \dot{\mathbf{v}}(t_2) = \mathbf{0}$, τ is any real number, and

$$\int_{t_1}^{t_2} \frac{d}{d\tau}[I(\mathbf{u}(t) + \tau\mathbf{v}(t))]|_{\tau=0} dt = 0, \tag{14}$$

for all such $\mathbf{v}$. It can be shown that if the displacement $\mathbf{u}$ satisfies equation (14) of Hamilton's principle, then it must also satisfy the following differential equations respectively in each cases discussed above. For the Euler rod,

$$\rho\frac{\partial^2 u}{\partial t^2} = K\frac{\partial}{\partial x}(|\frac{\partial u}{\partial x}|^{n-1}\frac{\partial u}{\partial x}) + f, \tag{15}$$

for the Euler beam

$$\rho A\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2}{\partial x^2}(KI_n|\frac{\partial^2 v}{\partial x^2}|^{n-1}\frac{\partial^2 v}{\partial x^2}) + f, \tag{16}$$

for the plane strain body

$$\rho\frac{\partial^2 u}{\partial t^2} = K[\frac{\partial\partial x}{(}I\partial u\partial x + \frac{\partial\partial y}{(}I\partial u\partial y] + f, \tag{17}$$

where$I = [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]^{\frac{n+-1}{2}}$ after a scaling of $\sqrt{2}$ in y.; and for the Plate

$$\rho h\frac{\partial^2 w}{\partial t^2} = D_n[\frac{\partial^2}{\partial x^2}(|D(w)|^{n-1}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})) + \frac{\partial^2}{\partial y^2}(|D(w)|^{n-1}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})) \tag{18}$$
$$+(1-\nu)\frac{\partial^2}{\partial x\partial y}(|D(w)|^{n-1}\frac{\partial^2 w}{\partial x\partial y})] - hf,$$

where D_n and $|D(w)|$ are given by (13).

4 Numerical simulation schemes

For simplicity, the above wave equations can be written in the abstract form $\rho\ddot{\mathbf{u}} = A(\mathbf{u})\mathbf{u} + \mathbf{f}$, which can be discretized by finite element and/or finite difference methods. The semi-discrete equation as a result of finite element method in spatial domain takes the form of a second order ODE in time $\mathbf{M}\ddot{\mathbf{X}} = \mathbf{K}(\mathbf{X})\mathbf{X} + \mathbf{F}$, where $\mathbf{M}$ is the mass matrix, $\mathbf{K}(\mathbf{X})$ is the nonlinear stiffness matrix, and $\mathbf{F}$ the load vector. Finite difference, such as Newmark, Runge-Kutta, or improved Euler schemes can be effectively used to simulate the waves for as solutions to these equations. Figure (1) is the displacement wave of (15) for

Acknowledgments

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